The Streaming Capacity of Sparsely-Connected P2P Systems with Distributed Control

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Abstract

Peer-to-Peer (P2P) streaming technologies can take advantage of the upload capacity of clients, and hence can scale to large content distribution networks with lower cost. A fundamental question for P2P streaming systems is the maximum streaming rate that all users can sustain. Prior works have studied the optimal streaming rate for a complete network, where every peer is assumed to be able to communicate with all other peers. This is however an impractical assumption in real systems. In this paper, we are interested in the achievable streaming rate when each peer can only connect to a small number of neighbors. We show that even with a random peer-selection algorithm and uniform rate allocation, as long as each peer maintains $\Omega(\log N)$ downstream neighbors, where N is the total number of peers in the system, the system can asymptotically achieve a streaming rate that is close to the optimal streaming rate of a complete network. These results reveal a number of important insights into the dynamics of the system, base on which we then design simple improved algorithms that can reduce the constant factor in front of the $\Omega(\log N)$ term, yet can achieve the same level of performance guarantee. Simulation results are provided to verify our analysis.

I. INTRODUCTION

With the proliferation of high-speed broadband services, the demand for rich multimedia content over the Internet, in particular high-quality video delivery over the Internet, has kept increasing. Streaming video directly from the server requires a large amount of upload bandwidth at the server, which can be very costly. The service quality can also be poor when the clients are far away from the server. In addition, it may be difficult for the server bandwidth to keep up when the demand is exceedingly high. There have been different approaches to off-load traffic from the server, using either CDN (content distribution network) or P2P (peer-to-peer) technologies. Deploying a large CDN can introduce a high fixed cost. In contrast, P2P technologies are particularly attractive because they take advantage of the upload bandwidth of the clients, which does not incur additional cost to the video service provider. Several well-known commercial P2P live streaming systems have been successfully deployed, include CoolStreaming [2],

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PPLIVE [3], TVAnts [4], UUSEE [5], PPStream [6]. A typical P2P streaming system can now offer thousands of TV channels or movies for viewing, and may serve hundreds of thousands of users simultaneously [5].

In contrast to the practical success of these P2P streaming systems, the theoretical understanding of the performance of P2P streaming seems to be lagging behind, which may impede further improvement of P2P live streaming. A basic question can be asked is what is the maximum streaming rate that all users can sustain for all possible policies? This question has been studied under the assumption of a complete network, where each peer can connect to all other peers simultaneously. Under this assumption, the maximum streaming capacity has been found in [7], and both centralized and distributed rate allocation algorithms to achieve this maximum streaming capacity have been developed [7]–[10]. However, the assumption of a complete network is impractical for any large-scale P2P streaming systems. In a real P2P streaming system, typically each peer is only given a small list of other peers (which we refer to as neighbors) chosen from the entire population, and each peer can only connect to this subset of neighboring peers (neighbors may not be close in terms of physical distance). The number of neighboring peers is often much smaller than the total population, in order to limit the control overhead.

When each peer only has a small number of neighbors, the P2P network can be modeled as an incomplete graph with node-degree constraints. In this case, the streaming capacity of P2P systems becomes more complicated to characterize. Liu et al. [11] investigate the case when the number of downstream peers in a single sub-stream tree is bounded. However, the number of neighbors that each peer could have over all sub-streams can still be very large (in the worse case it can be connected to all the other peers simultaneously). Some approximated and centralized solutions to solve the optimal streaming capacity problem on a given incomplete network has been proposed in [12]. However, for large-scale P2P streaming systems, such a centralized approach will be difficult to scale. Liu et al. [13] proposed a Cluster-Tree algorithm to construct a topology subject to a bounded node-degree constraint, which could achieve a streaming rate that is close to the optimal streaming capacity of a complete network. This result gives us hope that, even with node-degree constraints, a P2P network may achieve almost the same streaming rate as that of a complete network. However, the Cluster-Tree algorithm is not a completely de-centralized algorithm because it requires the tracker (a central entity) to apply the Bubble algorithm at the cluster level. The Bubble algorithm is a centralized algorithm. Some other works such as SplitStream [14] and Chinasaw [15] have also studied the problem of how to improve the streaming capacity when there is a node-degree constraint. However, these works did not provide theoretical results on the achievable streaming rate. To the best of our knowledge, there is no fully distributed algorithm in the literature that can achieve close-to-optimal P2P streaming capacity in incomplete networks.

In this paper, we are interested in the following question: without centralized control, how many neighbors does a peer in a large P2P network need to maintain in order to achieve a streaming capacity that is close to the optimal streaming capacity of an otherwise complete network? Further, can we develop fully-distributed algorithms for peer-selection and rate-allocation to achieve the close-to-optimal streaming capacity? This paper provides some interesting and positive answers to these questions. We first show that, if each peer has $\Omega(\log N)$ neighbors, where N is the total number of peers in the system, close-to-optimal streaming rate can be achieved with probability approaching 1 as N goes to infinity. Further, in order to achieve this goal, each peer only needs to choose $\Omega(\log N)$ downstream neighbors uniformly and randomly from the entire population, and simply allocates its upload capacity evenly among all downstream peers. Only the server needs a slightly different peer-selection policy (see Section II-B for details).

The results that we obtain have a similar flavor as scaling-law results in wireless ad hoc networks [16]. Although such results only hold when the size of the network N is large, they do provide important insights into the dynamics of the system. For example, our analysis indicates that, with a random peer selection strategy, for each user the most likely bottle neck for its streaming capacity is at the "last hop", i.e. the sum of the upload capacity allocated to this user by its immediate upstream neighbors. This insight suggests that we could focus on balancing the capacity at the *last* hop when designing new distributed resource allocation algorithms for P2P streaming. Based on this insight, we then design an alternative algorithm that can substantially reduce the number of neighbors required to achieve the same probability of attaining the near-optimal streaming rate. This improved algorithm is still very simple and can be implemented in a distributed fashion. Hence, we believe that the insights from these results can be very helpful for designing more efficient control algorithms for P2P streaming. Finally, although due to space constraints we focus in this paper on single-channel P2P systems (i.e., only one video is served), we believe that the results and insights obtained here can also be generalized to multi-channel P2P systems [17]. Readers can refer to [1] for examples.

II. SYSTEM MODEL AND MAIN RESULT

In this section, we will show that even without centralized control, $\Omega(\log N)$ neighbors are sufficient for large P2P streaming networks. Specifically, we will show that just by letting each peer select its $\Omega(\log N)$ neighbors randomly and do uniform rate allocation among these neighbors, the close-to-optimal streaming rate could be achieved with high probability when the network size N is large.

A. System Model

We consider a peer-to-peer live streaming network with N peers and one source s. In the rest of the paper, we will use the terms "source" and "server" interchangeably. Similarly, we will use the terms "peer", "node" and "user" interchangeably. Denote the set of all peers and the source as V (thus, |V| = N + 1). We assume that the source has an infinitely long video stream to be streamed to all peers and it has a fixed upload capacity u_s . Let U_i denote the upload capacity of peer *i*. For ease of exposition, we use a simple ON-OFF model to model the heterogeneity and random variation of the upload capacity: each peer has an upload capacity of $U_i = u$ with probability p and an upload capacity of $U_i = 0$ with probability 1 - p, i.i.d. across peers. Thus, an ON peer represents a user with large upload capacity, while an OFF peer represents a user with low upload capacity.¹ We assume that $u_s \ge u$.

¹We note that the ON-OFF model can be viewed as the most extreme case of heterogeneous upload capacity. In fact, among all possible distributions of the peers' upload capacity that are between $[0, U_{max}]$ and that have the same mean μ , the ON-OFF model has the largest variance. Hence, the uncertainty/variability of the ON-OFF model will be the largest, and the performance of the system will also likely be the worst. Based on this relationship, we can also generalize the main conclusions of this paper to other distributions for the upload capacity (see also the numerical results in Section V). Interested readers can refer to Section IV

Like other works [7], [12], [13], [18], we assume that the download capacity and the core network capacity are sufficiently large, and hence the only capacity constraints are on the upload capacity. Each peer $i \in V \setminus \{s\}$ has a fixed set \mathcal{E}_i of M downstream neighbors. Similarly, the source has a set \mathcal{E}_s of M downstream peers. We can then model the P2P network as a directed and capacitated random graph [19]. If $j \in \mathcal{E}_i$, assign a directed edge (i, j)from i to j. Let the set of all edges be E. Note that there may be multiple peers that have a common downstream neighbor. Define C_{ij} and C_{sj} be the streaming rate from peer i and source s, respectively, to peer j.

The values of \mathcal{E}_i , \mathcal{E}_s , C_{ij} and C_{sj} depend on the peer-selection and rate-allocation algorithm. Given such an algorithm, we can define the "streaming capacity" of the system as the maximum rate that the source can distribute the streaming content to all peers. For example, for a complete network, we have $\mathcal{E}_i = V \setminus \{i, s\}$ and $\mathcal{E}_s = V \setminus \{s\}$. Under such an idealized setting, [7] shows that the optimal streaming capacity is $\min \left\{ u_s, \frac{u_s + \sum_{i \in V} U_i}{N} \right\}$, and it can be achieved by setting $C_{ij} = U_i/(N-1)$ and $C_{sj} = U_s/N$ for all i, j. Note that the min(\cdot) function is a concave function. Therefore, the expectation of the above optimal streaming capacity satisfies

$$\mathbf{E}\left[\min\left\{u_{s}, \frac{u_{s} + \sum_{i \in V} U_{i}}{N}\right\}\right]$$

$$\leq \min\left\{u_{s}, \frac{u_{s} + \sum_{i \in V} \mathbf{E}[U_{i}]}{N}\right\} \triangleq C_{f}.$$
 (1)

For ease of exposition, we refer to C_f as "the optimal streaming capacity" throughout the rest of this paper. For our ON-OFF model of upload capacity, this optimal streaming capacity is equal to $C_f = \min \{u_s, \frac{u_s}{N} + up\}$. However, as we discussed in the introduction, the assumption of a complete network is impractical. In this paper, we are interested in the streaming capacity of an incomplete network, which can be calculated by the minimum cuts. Specifically note that for a given user t, a cut that separates s and t is defined by dividing the peers in V into a set V_n of size (n + 1) that contains the server, and the complementary set V_n^c of size (N - n) that contains the peer t, i.e.,

$$s \in V_n, |V_n| = n + 1, t \in V_n^c$$
 and $|V_n^c| = N - n$.

The capacity of the cut C_n is defined as $C_n = \sum_{i \in V_n} \sum_{j \in V_n^c} C_{ij}$. See Fig 1 for illustration.

Let $C_{\min}(s \to t)$ denote the minimum-cut capacity, which is the minimum capacity of all cuts that separate the source s and the destination t. It is well-known that this min-cut capacity is equal to the maximum rate from s to t. Let $C_{\min - \min}(s \to T)$ denote the min-min-cut which is the minimum cut of all individual min-cut capacities from the source to each destination t within a set T, i.e.,

$$C_{\min-\min}(s \to \mathcal{T}) = \min_{t \in \mathcal{T}} C_{\min}(s \to t).$$

The streaming capacity of the network is then equal to $C_{\min - \min}(s \to V \setminus \{s\})$ [20]. Note that given the graph and the capacity of each edge, this streaming capacity can be achieved with simple transmission schemes, e.g., with network coding [21], [22] or with a latest-useful-chunk policy [8]. However, it may required global knowledge and centralized control in order to optimally construct the network graph and allocate the upload capacity. A natural question is then the following: without centralized control, can the streaming capacity over an incomplete network



Fig. 1. Illustration of the neighbor selection and a cut

approach the optimal streaming capacity C_f of a complete network? In the next subsection we will provide a simple and distributed peer-selection and rate-allocation algorithm that can achieve this with high probability when the network size is large.

B. Algorithms

We will now give explicit description of our simple control algorithm. First, we use a random peer-selection algorithm. Specifically, each peer randomly selects M downstream neighbors uniformly from all other peers. On the other hand, the server selects M downstream neighbors uniformly and randomly among the ON peers. We note that uniformly-random peer-selection is very easy to implement in practice, even with dynamic peer arrivals and departures. Specifically, note that the number of upstream neighbors of a peer will be a binomial random variable (sum of N Bernoulli random variables with mean $\frac{M}{N}$). Note that the mean of X is M. Thus, when a new peer joins the system, it simply contacts X peers chosen uniformly randomly among the existing peers. Then, each contacted peer will choose one of its current downstream neighbor. Further, the new peer selects M downstream neighbors uniformly randomly, break this downstream neighbors uniformly randomly among the existing peers. On the other hand, when a peer leaves the system, all of its upstream neighbors simply re-selects a new downstream neighbor randomly. With this mechanism, it is easy to verify that, at any point in time, the set of M downstream neighbors of each peer is uniformly distributed among the current set of active peer.

Second, we use a uniform rate-allocation algorithm, i.e., each peer *i* simply divides its upload capacity equally among all of its downstream neighbors in \mathcal{E}_i . Therefore, each peer in the set \mathcal{E}_i receives a streaming rate U_i/M from peer *i*. Similarly, each downstream peer of the server receives U_s/M from the server. Under the above scheme, the link capacity C_{ij} is given by

$$C_{ij} = \begin{cases} U_i/M, & \text{if } j \in \mathcal{E}_i, i \neq s \\ U_s/M, & \text{if } j \in \mathcal{E}_s, i = s \\ 0, & \text{otherwise.} \end{cases}$$

Note that since \mathcal{E}_i and \mathcal{E}_s are chosen randomly, C_{ij} 's are also random variables. We define another import parameter for the total capacity that each peer *i* directly receives from its upstream neighbors, which is given by $C_i^R = \sum_{j \in V} C_{ji}$. We will see that this value is the main factor that determines the streaming capacity from the source to each node.

Remark: Since an OFF peer represents a user with low upload capacity, the above scheme implies that, regardless of each user's upload capacity, it will choose the same number M of downstream neighbors uniformly and divide its capacity evenly among these downstream neighbors. In Section IV, we use this model and show that, even with a general distribution of upload capacity, $O(\log N)$ neighbors are still sufficient to attain a close-to-optimal streaming capacity.

Somewhat surprisingly, we will show that, as long as $M = O(\log N)$, the algorithm achieves close-to-optimal streaming capacity, with probability approaching 1 as $N \to \infty$ (Theorem 1).

Remark: Note that the server only chooses ON peers as its downstream neighbors. This is essential for achieving the close-to-optimal streaming capacity. To see this, note that the optimal streaming capacity C_f of a complete network is also constrained by the server capacity (see Equation (1)). If the server had used a substantial fraction of its upload capacity to serve OFF peers, intuitively the rest of the peers would then suffer a lower streaming rate. With the same intuition, one would think that the peers directly connected to the server also need to be careful in choosing their downstream neighbors. However, this turns out to be unnecessary. For our main result (Theorem 1) to hold, no other peers (except the server) are required to differentiate their downstream neighbors. As readers will see, this is because those cuts with V_n only containing the downstream neighbors of s play a small role in the overall probability of attaining the close-to-optimal streaming capacity.

We also note that the above algorithm uses the "push" model, where upstream peers choose downstream neighbors. An alternate model is the "pull" model, where downstream peers choose upstream neighbors. Note that both models create a mesh-topology, and there is considerable symmetry between the two models. We use the push model in this paper because it is easier to analysis, although we believe that the main results of the paper can be generalized to the pull model, which we leave as future work.

C. Main Result

Theorem 1. For any $\epsilon \in (0,1)$ and d > 1, there exist α and N_0 such that for any $M = \alpha \log(N)$ and $N > N_0$ the probability for the min-min-cut under the algorithm in Section II-B to be smaller than $(1-\epsilon)C_f$ is bounded by

$$\mathbf{P}\left(C_{\min-\min}(s\to V)\leq (1-\epsilon)C_f\right)\leq O\left(\frac{1}{N^{2d-1}}\right).$$

Recall that the min-min-cut is equal to the streaming rate to all peers. Hence, Theorem 1 shows that as long as the number of downstream neighbors M is $\Omega(\log N)$, for any $\epsilon \in (0, 1)$ the streaming rate of our algorithm will be larger than $(1 - \epsilon)$ times the optimal streaming capacity with probability approaching 1 as the network size N increases.

D. Proof of Theorem 1

We first find the min-cut for any fixed peer t. We will use a similar approach as the one in [19]. We will show that the probability for the capacity of a cut to be smaller than $(1 - \epsilon)$ times its mean is very small, as N becomes large. Then, we will take the union bound over all cuts and show that overall probability is also very small. However, the techniques in [19] do not directly apply to our model due to the following two reasons. First, due to the ON-OFF model, there are fewer "ON" peers and hence the probability for each cut to fall below its expected value is larger than the case when all peers' upload capacity is the same. However, there are still the same number of cuts we need to account for, which may cause the union bound in [19] to diverge. Second, the link capacity C_{ij} in [19] is assumed to be independent across j, which is not the case in our model. To address the first difficulty, we will first consider the subgraph that only contains the ON users, and hence the number of cuts is also reduced correspondingly. To address the second difficulty, we will show that the joint distribution of C_{ij} can be approximated by i.i.d. random variables, which significantly simplifies the analysis.

We first introduce the following general relationship between the min-cut from the server s to the peer t in a random graph G and the min-cut from the server s to the peer t in the any subgraph H_t of G that contains s and t.

Proposition 2. Let G be a random graph defined on some probability space Ω that has a fixed source s and a fixed destination t. Let H_t be another random graph defined on the same probability space such that $H_t(\omega) \subseteq G(\omega)$ for all $\omega \in \Omega$ and H_t contains s and t. Then for any given positive value C, the following holds,

$$\mathbf{P}\left(C_{\min,G}(s \to t) \le C\right) \le \mathbf{P}\left(C_{\min,H_t}(s \to t) \le C\right).$$
(2)

where $C_{\min,G}(s \to t)$ is the min-cut in G from s to t, and $C_{\min,H_t}(s \to t)$ is the min-cut in H_t from s to t.

Proof: Let $\mathcal{A} = \{G(\omega) : C_{\min,G(\omega)}(s \to t) \leq C\}$ and $\mathcal{B} = \{\omega : C_{\min,H_t(\omega)}(s \to t) \leq C\}$. For any $\omega \in \mathcal{A}$, the min-cut from s to t in the graph $G(\omega)$ is less than C. Since H_t is a subgraph of $G(\omega)$, the min-cut from s to t in $H_t(\omega)$ is smaller than the min-cut in G, i.e., $C_{\min,H_t(\omega)}(s \to t) \leq C_{\min,G(\omega)}(s \to t) \leq C$. Hence, $\omega \in \mathcal{B}$. We then have $\mathcal{A} \subseteq \mathcal{B}$ and (2) holds consequently.

Proposition 2 is intuitive because every cut in $G(\omega)$ has a larger capacity than the corresponding cut in the subgraph $H_t(\omega)$. For a given destination t, let $H_t(W, F)$ be the subgraph of G(V, E) such that W contains the peer t, the server and all of the nodes whose channel condition is ON, and $F \subset E$ are those edges between nodes in W. The capacity of the edges in F is the same as the capacity of the edges in E. Proposition 2 allows us to focus on the subnetwork H_t instead of the entire network G. Assume that there are Y ON peers in the network excluding peer t, and thus |W| = Y + 2. Clearly, Y is a random variable with binomial distribution with parameter N - 1 and p. For ease of exposition, we assume that Y is fixed during the following discussion for one given cut, and we will consider the randomness of Y later when we take the union bound over all cuts. We define a cut on

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 H_t by dividing the peers in W into a set W_m of size m + 1 that contains the server, and the complementary set W_m^c of size Y - m + 1 that contains peer t. The capacity of the cut D_m is then given by

$$D_m = \sum_{k \in W_m^c} C_{si} + \sum_{i \in W_m} \sum_{k \in W_m^c} C_{ik}.$$
(3)

Note that for each peer $i \in W_m$ (and $i \neq s$), we have $\sum_{k \in W_m^c} C_{ik} = L_i u/M$, where L_i is the number of downstream neighbors of peer i that are in the set W_m^c . Note that the value of L_i must satisfy $\max\{0, M - (N - Y + m - 2)\} \leq L_i \leq \min\{M, Y - m + 1\}$. Since downstream neighbors of peer i are uniformly chosen from other peers, we have

$$\mathbf{P}\left(\sum_{k\in W_m^c} C_{ik} = l \cdot \frac{u}{M}\right) = \frac{\binom{Y-m+1}{l}\binom{N-Y+m-2}{M-l}}{\binom{N-1}{M}}.$$

This is the probability that l out of M downstream neighbors of peer i are in W_m^c (of size Y - m + 1) and M - l of them are in the set W_m . The distribution of L_i is known as a hyper-geometric distribution with expectation $\frac{(Y-m+1)M}{N-1}$ [23, p167]. We can get a similar expression for the source s, i.e.,

$$\mathbf{P}\left(\sum_{i\in W_m^c} C_{si} = l \cdot \frac{u_s}{M}\right) = \begin{cases} \frac{\binom{Y-m}{l}\binom{m}{M-l}}{\binom{Y}{M}} & \text{if } t \text{ is OFF} \\ \frac{\binom{Y-m+1}{l}\binom{m}{M-l}}{\binom{Y+1}{M}} & \text{if } t \text{ is ON.} \end{cases}$$
$$\mathbf{E}\left[\sum_{i\in W_m^c} C_{si}\right] = \begin{cases} \frac{u_s(Y-m)}{Y} & \text{if } t \text{ is OFF}, \\ \frac{u_s(Y+1-m)}{Y+1} & \text{if } t \text{ is ON.} \end{cases}.$$

Hence, we obtain the expectation of D_m as

$$\mathbf{E}\left[D_{m}\right] = \mathbf{E}\left[\sum_{k \in W_{m}^{c}} C_{si}\right] + \sum_{i \in W_{m}} \mathbf{E}\left[\sum_{k \in W_{m}^{c}} C_{ik}\right]$$
$$= \begin{cases} \frac{u_{s}(Y-m)}{Y} + \frac{u}{N-1}m(Y-m+1) & \text{if } t \text{ is OFF,} \\ \frac{u_{s}(Y+1-m)}{Y+1} + \frac{u}{N-1}m(Y-m+1) & \text{if } t \text{ is ON.} \end{cases}$$
(4)

Next, we are interested in the probability that $D_m \ge (1 - \epsilon)\mathbf{E}[D_m]$ for all m for a given constant $\epsilon \in (0, 1)$. In other words, this is the probability that the min-cut value is no less than $(1 - \epsilon)$ times its average. For all m, it is not hard to see

$$\mathbf{E}[D_m] \ge \min\{\mathbf{E}[D_0], \mathbf{E}[D_Y]\} = \min\left\{u_s, \frac{u_s}{Y} + \frac{Y}{N-1}u\right\}.$$

If we have $Y \ge (1 - \epsilon)p(N - 1)$, we will get

$$\mathbf{E}[D_m] \ge (1-\epsilon) \min\left\{u_s, \frac{u_s}{N} + pu\right\}.$$

Recall that $C_f = \min\{u_s, \frac{u_s}{N} + pu\}$ is the optimal streaming capacity assuming a complete network [7]. Hence, $D_m \ge (1-\epsilon)\mathbf{E}[D_m]$ then implies that $D_m \ge (1-\epsilon)^2 C_f$. In other words, the probability that $D_m \ge (1-\epsilon)\mathbf{E}[D_m]$ for all m becomes a lower bound for the probability that the min-cut is no less than $(1-\epsilon)^2 C_f$. In the following, we will derive $\mathbf{P}(D_m \ge (1-\epsilon)\mathbf{E}[D_m])$. We will use the moment generating function for D_m . Before we go further, we need to address the second difficulty we mentioned above, i.e., the C_{ij} 's are correlated across j. To remove the coupling, we need to introduce the notion of negatively related for Bernoulli random variables [24], [25].

Definition 3. The Bernoulli random variables I_i , i = 1, ..., n, are said to be negatively related if for each $i \le n$ there exists random variables J_{ij} , such that the distribution of the random vector $[J_{i1}, J_{i2}, ..., J_{in}]$ is equal to the distribution of the random vector $[I_1, I_2, ..., I_n]$, given that $I_i = 1$ and $J_{ij} \le I_j$ for $j \ne i$.

For negatively related random variables, the following theorem holds (Theorem 4 in [25]).

Theorem 4. Suppose I_i 's are negatively related Bernoulli random variables with identical distribution, i = 1, 2, ..., n. Let \tilde{I}_i , i = 1, 2, ..., n, be i.i.d. random variables, where \tilde{I}_i has the same distribution as I_i for all i. Then for any real t,

$$\mathbf{E}\left[e^{t\sum_{i=1}^{n}I_{i}}\right] \leq \mathbf{E}\left[e^{t\sum_{i=1}^{n}\tilde{I}_{i}}\right].$$

Theorem 4 thus allows us to bound the moment generating function of negatively related random variables by that of independent random variables. Its intuition can be explained as follows. Roughly speaking, for negatively related Bernoulli random variables, conditioned on the event that one of them is 1, the others are more likely to be small. Correspondingly, conditioned on the event that one of them is 0, the others are more likely to be large. Therefore, when t > 0, the moment generating function is mainly determined by the probability of the sum of all indicator random variables achieving the larger value. The sum of negatively related random variables is less likely to achieve a larger value and hence the value of the moment generation function is smaller. For t < 0, the moment generating the sum of all indicator random variables achieving the probability of the sum of all indicator random variables achieving the probability of the sum of all indicator random variables achieving the probability of the sum of all indicator random variables achieving the probability of the sum of all indicator random variables achieving the probability of the sum of all indicator random variables achieving the probability of the sum of all indicator random variables achieving the smaller value. The sum of negatively related random variables is also less likely to achieve a smaller value and hence the value of the moment generation function is smaller.

One can show that hyper-geometric random variables can be viewed as the sum of negatively related Bernoulli random variables (See Example 1 in [25]). Specifically, we first construct I_i by choosing M neighbors out of N-1 peers. For each peer i on the right, let $I_i = 1$ if peer i is chosen as a neighbor, and let $I_i = 0$ otherwise (Note that I_i is not defined for peers on the left). We can then construct J_{ij} as follows. First, set $J_{ij} = I_j$ for all j. Then if $J_{ii} = 0$, in order to make $J_{ii} = 1$, we choose one neighbor k randomly (either from the left or the right), and exchange that neighbor with peer i. If k was on the left, we then let $J_{ii} = 1$. If k was on the right, we then let $J_{ii} = 1$ and $J_{ik} = 0$. Clearly, J_i has the same distribution as I given that $I_i = 1$. However, by our construction, $J_{ij} \leq I_j$ for all $j \neq i$. Hence, $I_i, i = 1, ..., M$, are negatively related. We can now use Theorem 4 to bound the moment generation function of $\sum_{k \in W_m^d} C_{ik}$ by the moment generating functions of the sum of i.i.d. random variables. Towards this end, we have the following Proposition.

Proposition 5. For any given cut V_k and V_k^c of a network G(V, E), let \tilde{W}_1 and \tilde{W}_2 be subsets of V_k and V_k^c , respectively. Assume that $|\tilde{W}_1| = q \le k + 1$ and $|\tilde{W}_2| = r \le N - k$. Let the upload capacity of each peer $i \in \tilde{W}_1$

be u. For each peer in \tilde{W}_1 , it chooses M downstream neighbors uniformly and randomly from a given subset \tilde{V} of V that is a superset of \tilde{W}_2 . Let $\tilde{N} = |\tilde{V}|$. Then the moment generating function of $\sum_{i \in \tilde{W}_1} \sum_{j \in \tilde{W}_2} C_{ij}$ satisfies

$$\mathbf{E}\left[e^{-\theta\sum_{i\in\tilde{W}_{1}}\sum_{j\in\tilde{W}_{2}}C_{ij}}\right] \le \exp\left[Mq\frac{r}{\tilde{N}}\left(e^{-\theta\frac{u}{M}}-1\right)\right].$$
(5)

Note that the right hand side of (5) can be viewed as the moment generating function of $\sum_{i \in \tilde{W}_1} \sum_{j \in \tilde{W}_2} C_{ij}$ assuming that C_{ij} 's are independent. The proof of Proposition 5 can be found in Appendix B. Proposition 5 combined with the Chernoff bound will be frequently used to estimate the probability for a cut to "fail", i.e., the capacity of a cut being less than $(1 - \epsilon)$ times its expected capacity. Recall that the capacity D_m of the cut W_m is given by (3). Then by taking \tilde{W}_1 and \tilde{W}_2 in Proposition 5 to be W_m and W_m^c , respectively, we can show the following result for the cut W_m in H_t under the assumption of ON-OFF upload capacities.

Lemma 6. Let $\epsilon \in (0, 1)$. Given that the total number of ON peers in the entire network Y is equal to y, the probability that the capacity D_m of the cut W_m in H_t is less than $(1 - \epsilon)\mathbf{E}[D_m]$ can be bounded by the following,

$$\mathbf{P}(D_m \le (1-\epsilon)\mathbf{E}[D_m]|Y=y)$$

$$\le \exp\left[-\left(Mm\frac{y-m+1}{N-1} + M\frac{y-m}{y}\right)\frac{u}{u_s}\frac{\epsilon^2}{2}\right]$$

The proof of Lemma 6 can be found in Appendix C. Lemma 6 gives us an upper bound on the probability that the capacity D_m of a cut W_m is less than $1 - \epsilon$ times its mean conditioned on the event that the total number of ON peers Y is equal to y. Note that $Mm\frac{y-m+1}{N}$ is the average number of edges from peers in W_m to peers in W_m^c , while $M\frac{y-m}{y}$ is a lower bound on the average number of edges from the server to peers in W_m^c . Hence, the upper bound in Lemma 6 decreases exponentially if the average number of edges increases. Furthermore, since the average number of edges is proportional to M, the upper bound also decreases exponentially if M increases. We will use Lemma 6 for each m = 1, 2, ..., Y. The following lemma then bounds the effect of all cuts separating s and t. Note that for each value of m, there are $\binom{Y}{m}$ possible cuts W_m . Due to symmetry, the capacity of all $\binom{Y}{m}$ cuts has the same distribution.

Lemma 7. Define $\tilde{\mathcal{B}}_m$ to be the event $\{D_m \leq (1-\epsilon)C_f \text{ for any cut } W_m \text{ among the } \binom{Y}{m} \text{ cuts } \}$. Suppose that there exists $\eta \in (0,1)$ such that for any $y \geq \eta pN$ and any integer m between 0 and y, the following holds for $\beta = \exp(-M\frac{u}{u_s}\frac{\epsilon'^2}{2})$ and $\gamma = \eta p$,

$$\mathbf{P}(D_m \le (1-\epsilon)C_f | Y=y) \le \beta^m \frac{y-m+1}{N-1} + \frac{y-m}{y}.$$

Then, the probability of the union of all $\tilde{\mathcal{B}_m}$'s is bounded by

$$\mathbf{P}\left(\bigcup_{m=0}^{Y}\tilde{\mathcal{B}}_{m}\right) \leq O(e^{-(1-\eta)^{2}p^{2}N}) + \beta^{\gamma}\left[\left(1+p\beta^{\frac{\gamma}{2}}\right)^{N-1}\right].$$

In addition, we can separate the union bound into two parts:

$$\mathbf{P}\left(\bigcup_{m=0}^{Y-1} \tilde{\mathcal{B}}_{m}\right) \leq O(\exp(-(1-\eta)^{2} p^{2} N)) \\
+ \beta^{\gamma} \left[\left(1+p\beta^{\frac{\gamma}{2}}\right)^{N-1}-1\right],$$
(6)

$$\mathbf{P}\left(\tilde{\mathcal{B}}_{Y}\right) \le O(\exp(-(1-\eta)^{2}p^{2}N)) + \beta^{\gamma}.$$
(7)

Lemma 7 is obtained by taking the union bound over all cuts. The detailed proof of Lemma 7 is in Appendix D. Combing Lemma 6 and Lemma 7, we can now prove Theorem 1.

Proof of Theorem 1: According to Proposition 2 and Lemma 7, for any peer t, the minimum cut from the source s to t can be bounded by

$$\mathbf{P}\left(C_{\min}(s \to t) \le (1 - \epsilon)C_f\right)$$
$$\leq \mathbf{P}\left(C_{\min,H_t}(s \to t) \le (1 - \epsilon)C_f\right) = \mathbf{P}\left(\bigcup_{m=0}^{Y} \tilde{\mathcal{B}}_m\right).$$
(8)

Recall that if $Y \ge \sqrt{1-\epsilon}pN$, $D_m \ge \sqrt{1-\epsilon}\mathbf{E}[D_m]$ implies $D_m \ge (1-\epsilon)C_f$. By Lemma 6, letting $\epsilon' = 1-\sqrt{1-\epsilon}$ and $\beta = \exp(-M\frac{u}{u_s}\frac{\epsilon'^2}{2})$, we have if $Y \ge (1-\epsilon')pN$,

$$\mathbf{P}(D_m \ge (1-\epsilon)C_f) \le \mathbf{P}(D_m \ge (1-\epsilon')\mathbf{E}[D_m])$$
$$\le \exp\left[-\left(Mm\frac{y-m+1}{N-1} + M\frac{y-m}{y}\right)\frac{u}{u_s}\frac{\epsilon'^2}{2}\right]$$
$$=\beta^{m\frac{y-m+1}{N-1} + \frac{y-m}{y}}.$$

Now let $\eta = 1 - \epsilon'$ and apply Lemma 7 to (8). We get

$$\mathbf{P}\left(C_{\min}(s \to t) \le (1 - \epsilon)C_f\right) \le \mathbf{P}\left(\bigcup_{m=0}^{Y} \tilde{\mathcal{B}}_m\right)$$
$$\le 2\beta^{\gamma} \left(1 + p\beta^{\frac{\gamma}{2}}\right)^{N-1} + O(\exp(-\epsilon'^2 p^2 N)).$$

Note that by assumption, $M = \alpha \log(N)$. For any $\epsilon > 0$ and $\epsilon' = 1 - \sqrt{1 - \epsilon}$, choose a sufficiently large α such that $\alpha \ge \frac{4du_s}{\gamma u \epsilon'^2}$. We then have, for large N,

$$\beta^{\gamma} = \exp(-M\gamma \frac{u}{u_s} \frac{\epsilon'^2}{2}) = \exp(-2d\log(N)) = 1/N^{2d}.$$

Hence, the minimum cut satisfies,

$$\mathbf{P}\left(C_{\min}(s \to t) \le (1 - \epsilon')C_f\right)$$
$$\le \frac{1}{N^{2d}} 2\left(1 + pO(\frac{1}{N^d})\right)^{N-1} = O\left(\frac{1}{N^{2d}}\right).$$

Thus, the min-min cut satisfies

$$\mathbf{P} \left(C_{\min - \min} \leq (1 - \epsilon) C_f \right)$$

$$\leq \sum_{t=1}^{N} \mathbf{P} \left(C_{\min}(s \to t) \leq (1 - \epsilon) C_f \right)$$

$$\leq O \left(\frac{1}{N^{2d}} \right) \cdot N = O \left(\frac{1}{N^{2d-1}} \right).$$

We remark on several implications of Theorem 1. First, Theorem 1 not only shows that pure random selection is sufficient to achieve close-to-optimal streaming capacity as long as each peer has $\Omega(\log N)$ downstream neighbors, it also reveals important insights on the significance of different types of cuts. To see this, note that if we choose α as in the proof such that $\beta^{\gamma} = O(1/N^{2d})$, we have (from (6))

$$\mathbf{P}\left(\bigcup_{m=0}^{Y-1}\tilde{\mathcal{B}}_{m}\right) \leq 2\beta^{\gamma} \left[\left(1+p\beta^{\frac{\gamma}{2}}\right)^{N-1}-1\right]$$
$$=O(1/N^{2d})O(e^{1/N^{d-1}}-1)=o(1/N^{2d}).$$

On the other hand, we have $\mathbf{P}\left(\tilde{\mathcal{B}}_Y\right) = O(1/N^{2d})$. Hence, the probability that the last cut (the W_Y and W_Y^c cut) fails is much larger than the probability that any other cut fails. Thus, for each peer t, the min-cut from the source to t is mainly determined by C_t^R (recall that C_t^R is the total capacity received by peer t directly from its upstream neighbors, which is also the capacity of the last cut).

The above insight suggests that, if we want to design improved distributed control algorithms for P2P streaming systems, we may want to focus on improving the capacity C_t^R at the last hop. Note that one of the main reasons for C_t^R to fall below its mean value is the imbalance of C_t^R across t. More specifically, some peers t may have a larger number of upstream peers, and hence have a larger-than-average value of C_t^R , while other peers may have a smaller-than-average value of C_t^R . Such imbalance will lead to an increase in the probability that some peers have low streaming rates. Based on this intuition, we can design a slightly more sophisticated scheme to balance the value of C_t^R of different peers, which will be discussed explicitly in section III.

Theorem 1 also reveals important relationships between the number of neighbors required and key system parameters. For example, if we require a better performance (smaller ϵ or larger d) or have fewer ON peers (smaller p), the number of downstream neighbors needed by each peer will increase. Specifically, according to the proof, we need $\alpha \geq \frac{4du_s}{\gamma u \epsilon'^2}$. If we require a higher streaming rate or a faster convergence rate, i.e., ϵ is smaller (consequently ϵ' is smaller) or d is larger, we will need a larger α . If the probability that a peer is ON is reduced, i.e., p is reduced, we will also need a larger α .

III. AN IMPROVED HYBRID ALGORITHM

In the previous section, we proposed a simple scheme with random neighbor selection and uniform rate allocation that can sustain a close-to-optimal streaming rate for all users. Our scheme only requires $O(\log N)$ neighbors for each peer. However, our simulation results (see Section V) indicate that the number of neighbors that each peer needs may still be quite large. This is because the actual number of neighbors required also depends on the constant factor α before the log N term. As in the remarks following Theorem 1, for uniform rate-allocation schemes, we

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factor α before the log N term. As in the remarks following Theorem 1, for uniform rate-allocation schemes, we need $\alpha \geq \frac{4du_s}{\gamma u \epsilon'^2}$, which increases inversely proportional to the square of ϵ' . The goal of this section is to study whether we can design a slightly more sophisticated scheme for neighbor selection and/or rate allocation that can significantly reduce the constant factor α . Specifically, our strategy is to retain the random peer-selection algorithm but focus on improving the rate allocation algorithm. One may argue that random peer-selection may still be suboptimal. However, as we explain in Section II-B, random peer-selection has the advantage that it is very easy to implement and robust to peer dynamics. In contrast, other peer-selection algorithms (e.g., based on forming tree [13]) will likely be more costly in the presence of peer dynamics. Since our goal in this paper is both to attain a close-to-optimal streaming capacity and to use simple, robust and distributed control, we believe that the choice of using random peer-selection strikes a reasonable trade-off. In fact, as we will show below, even by improving the rate-allocation alone, significant performance improvement can be attained.

As we observed in earlier sections, with high probability, the bottle neck for uniform rate allocation lies in the last hop, i.e., the total upload capacity allocated to some peers from their immediate upstream neighbors is smaller than average. Hence, a natural idea is to design a more sophisticated rate-allocation scheme such that the capacity of the last hop is more balanced, and therefore, we may be able to reduce the number of neighbors that each user needs in order to achieve a close-to-optimal streaming rate. More specifically, we may find $C_{ij} \ge 0$, $i, j \in V$, such that with as few neighbors as possible, the following holds

$$\sum_{j \in \mathcal{E}_i} C_{ij} \le u_i \text{ for all } i,$$

$$\sum_{i \in \mathcal{U}_j} C_{ij} \ge R_j \text{ for all } j,$$
(9)

where U_j denotes the set of all the upstream neighbors of peer *j*. Such a rate-allocation scheme is in general not difficult to complete: It can be found by solving a linear optimization problem. Wu and Li [26] has proposed a fully distributed rate-allocation algorithm to solve a similar linear program. However, the limitation of this approach is that such a rate-allocation scheme only guarantees the capacity for the last hop, and there may be another cut with smaller capacity, which still constrains the overall streaming rate of the system (readers can refer to Fig 3 for simulation results that confirm this observation.) On the other hand, if we were to formulate the rate-allocation problem as another linear program for the minimum cut, the complexity would be much higher than (9). Hence, it remains a challenging question to develop low-complexity rate-allocation algorithms that significantly outperform the uniform rate-allocation scheme.

Recall that in the previous section, using uniform rate allocation among the downstream neighbors, we show that all the other cuts have a much higher probability than the last-hop cut does, to achieve a larger rate than the required streaming rate. A natural question is then whether we can design a scheme that combines the advantages of both the more sophisticated rate allocation in (9) for improving the last-cut, and the uniform rate allocation for maintaining the high values at other cuts. This question leads us to the following hybrid algorithm that is simple to implement and significantly reduces the number of neighbors required.

We consider the following class of hybrid algorithms π_{θ} for rate allocation: each peer reserves a fraction $\theta \in (0, 1)$ of its upload capacity for the more sophisticated rate allocation similar to (9) and uses the remaining $(1-\theta)$ fraction of its upload capacity for uniform rate allocation. Specifically, let C_{ij}^S be the allocated capacity to j from i's θ fraction of upload capacity using the more sophisticated rate-allocation scheme, and let C_{ij}^U be the uniformly allocated capacity to peer j from peer i' remaining $(1-\theta)$ fraction of upload capacity. Note that each peer still randomly selects M downstream neighbors. Hence $C_{ij}^U = \frac{(1-\theta)u_i}{M}$ if $j \in \mathcal{E}(i)$. Then, the total allocated capacity from i to j is $C_{ij} = C_{ij}^U + C_{ij}^S = \frac{(1-\theta)u_i}{M} + C_{ij}^S$. We now formulate a linear feasibility problem to control C_{ij}^S . As we did before, we wish our algorithm could achieve a close-to-optimal streaming capacity. Hence we set the target streaming rate of each user j to be $R_j = (1 - \epsilon)C_f$. Recall that C_f is the optimal streaming capacity. Therefore, the goal of the more sophisticated rate allocation algorithm is to find C_{ij}^S 's such that

$$\sum_{\substack{j \in \mathcal{E}(i) \\ i \in \mathcal{U}(j)}} C_{ij}^{S} \le \theta u_{i}, \text{ for all } i,$$

$$\sum_{i \in \mathcal{U}(j)} \left(C_{ij}^{U} + C_{ij}^{S} \right) \ge (1 - \epsilon) C_{f}, \text{ for all } j.$$
(10)

Note that the distributed algorithm proposed in [26] is still suitable for solving this problem whenever the solution exists. Therefore, this hybrid algorithm still preserves the feature of being fully distributed and simple to implement. Next, we will show that it can achieve a close-to-optimal streaming capacity with a significantly lower number of neighbors.

A. Performance Analysis

Next we will show that this hybrid algorithm can achieve a streaming capacity of $(1 - \epsilon)C_f$ with a much smaller number of downstream neighbors of each peer. The following theorem states the performance of this hybrid algorithm more clearly.

Theorem 8. For any $\epsilon \in (0, 1)$, $\theta > 1/2$ and d > 1, there exist

$$\alpha \ge \max\left\{\frac{(2d)u_s}{\eta pu\max\left\{\frac{\epsilon^2}{2}, \frac{(2\theta-1)^2}{8\theta^2}\right\}}, \frac{2+\frac{p+\epsilon}{\theta}+d}{[p-\frac{(p+\epsilon)\delta}{\theta}](1-\epsilon)}\right\},\tag{11}$$

and N_0 such that for any $N > N_0$ and $M = \alpha \log(N)$, the probability that for the capacity of the min-min-cut under the algorithm π_{θ} is smaller than $(1 - \epsilon)C_f$ is bounded by

$$\mathbf{P}\left(C_{\min-\min}(s \to V) \le (1-\epsilon)C_f\right) \le O\left(\frac{1}{N^d}\right).$$

This result shows that the hybrid algorithm indeed reduces the lower bound on the number of required neighbors of each peer. Note that for small ϵ , the factor α does not depend on ϵ at all. In contrast, the factor α for the uniform rate-allocation scheme must increase proportional to $1/\epsilon^2$. As a numerical example, suppose that we want to sustain at least 90% of the optimal streaming capacity, which means that $\epsilon = 0.1$. The uniform rate-allocation scheme requires $\alpha \geq \frac{400 du_s}{up}$. In contrast, if we use the hybrid algorithm π_{θ} and choose $\theta = 0.9$, then we only need $\alpha \geq \frac{20 du_s}{up}$. The number of neighbors of each peers is reduced by 20 times.

We separate the proof of Theorem 8 into two parts. First, since the allocation of C_{ij}^S is based on (10), we need to show that, given the uniform rate allocation of $C_{ij}^U = \frac{(1-\theta)u_i}{M}$, there exists a feasible solution to (10) with high probability. Hence, all last cuts should be able to exceed the required streaming rate with high probability. Second, we need to show that, based on the uniform rate allocation C_{ij}^S alone, the values of all other cuts should also exceed the required streaming rate with high probability. Theorem 8 would then follows.

For the first step, we will use the following results, which state an equivalent characterization to (9) and (10). Specifically, there exists a rate-allocation such that the sum of the upload capacity allocated to each user from its immediate upstream neighbors is larger than its required streaming rate R_j if and only if, for any group of peers in the network, the total upload capacity from their upstream neighbors is larger than the sum of the streaming rates of this group of users.

Lemma 9. There exist $C_{ij} \ge 0$, $i, j \in V$, such that (9) holds if and only if for any subset $S \subseteq V$, the following holds

$$\sum_{i \in \mathcal{U}(\mathcal{S})} u_i \ge \sum_{j \in \mathcal{S}} R_j,\tag{12}$$

where $\mathcal{U}(\mathcal{S}) = \bigcup_{j \in \mathcal{S}} \mathcal{U}(j)$.

Corollary 10. There exist $C_{ij} \ge 0$, $i, j \in V$ such that (10) holds if and only if for any subset $S \subseteq V$, the following holds

$$\sum_{i \in \mathcal{U}(S)} \theta u_i \ge \sum_{j \in S} \left((1 - \epsilon) C_f - \sum_{i \in \mathcal{U}(j)} C_{ij}^U \right),$$
(13)

where $C_{ij}^U = \frac{(1-\theta)u_i}{M}$.

The proof of Lemma 9 follows a similar line of the argument as the Hall's Theorem [27]. The complete proof using the min-cut max-flow theorem is provided in Appendix A. Note that for the hybrid schemes, the reserved upload capacity of each user for the more sophisticated rate-allocation is θu_i . In addition, each user receives a capacity of $\sum_{i \in \mathcal{U}(j)} C_{ij}^U$ from the uniform rate allocation. Thus, since the required streaming rate for each user j is $(1-\epsilon)C_f$, the target downloading rate for the more sophisticated rate-allocation should be $(1-\epsilon)C_f - \sum_{i \in \mathcal{U}(j)} C_{ij}^U$. Therefore, Corollary 10 follows from Lemma 9 immediately, by letting the upload capacity of each user in Lemma 9 be θu_i , and letting R_j in Lemma 9 be $(1-\epsilon)C_f - \sum_{i \in \mathcal{U}(j)} C_{ij}^U$. Corollary 10 states that if (13) holds, then we can find a proper hybrid rate-allocation scheme such that the capacity of the last hop of each user is enough for its streaming rate. Next we will show that (13) holds with high probability. **Lemma 11.** Fix $\theta \in (0,1)$. For any $\epsilon \in (0,1)$ and d > 1, there exist N_0 and α_0 such that if $N \ge N_0$ and

$$\alpha \ge \frac{2 + (p + \epsilon)/\theta + d}{[p - (p + \epsilon)\delta/\theta](1 - \epsilon)},\tag{14}$$

the following holds for the hybrid algorithm π_{θ}

$$\begin{split} \mathbf{P}\left(\theta\sum_{i\in\mathcal{U}(\mathcal{S})}u_{i} \leq \sum_{j\in\mathcal{S}}\left[(1-\epsilon)C_{f} - \sum_{i\in V}C_{ij}^{U}\right] \text{ for some } \mathcal{S}\subset V\right)\\ \leq O\left(\frac{1}{N^{d}}\right). \end{split}$$

Lemma 11 and Corollary 10 together imply that the probability with which (10) has no solution, converges to 0 as the network size N grows. Therefore, with high probability, we can find a rate-allocation such that (10) holds, i.e., the capacities of all last-hop cuts are greater than $(1 - \epsilon)C_f$ with high probability. For others cuts, our random graph approach in Section II still applies. Theorem 8 then follows. Readers can refer to Appendix F for the detailed proof.

IV. GENERAL DISTRIBUTION FOR THE UPLOAD CAPACITY

In the previous sections, we have assumed that the upload capacity of each peer is either 0 or u. In this section, we are going to extend our result to the case when the random upload capacity of each peer follows a general distribution. Specifically, we now assume that the upload capacity U_i of user i is a bounded random variable with a general distribution, and is i.i.d. across i. Let U_{max} be the upper bound of all U_i 's, i.e., $0 \le U_i \le U_{\text{max}}$ for all $i \in V$. Assume that $\mathbf{E}[U_i] = \mu$, and let p be the probability that the upload capacity of a user is larger than its mean value, i.e., $\mathbf{P}(U_i \ge \mu) = p$. The following theorem holds.

Theorem 12. For any $\epsilon \in (0, 1)$, and d > 1, there exists α and N_0 such that for any $N > N_0$ and $M = \alpha \log(N)$, the probability for the min-min-cut to be smaller than $(1 - \epsilon)C_f$ is bounded by

$$\mathbf{P}\left(C_{\min-\min}(s \to V) \le (1-\epsilon)C_f\right) \le O\left(\frac{1}{N^d}\right).$$

Although the statement of this theorem is the same as Theorem 1, here we do not restrict the distribution of the user upload capacity to any specific distribution. To develop the proof of Theorem 12, consider the peers whose upload capacity U_i is larger than its mean value μ . These peers can be interpreted as "ON" peers. According to Proposition 2, we would hope that if the capacity from these peers alone is sufficiently large, we can just focus on the sub-network that only consists of these "ON" peers. Unfortunately, the capacity from the "ON" peers alone is not always enough. For example, fix a peer t, and consider the total capacity that this peer received from its direct upstream neighbors C_t^R , which is the capacity at the last hop as aforementioned. In the sub-network that only consists ON peers, the mean value of C_t^R can be calculated as

$$\mathbf{E}[C_t^R] = \frac{u_s}{N} + Mp\frac{\mu}{M} = \frac{u_s}{N} + p\mu < C_f.$$

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Hence, in the worst case, the mean value of C_t^R could be even smaller than C_f . Thus, there will be a significantly large probability that the actual value of C_t^R is less than C_f .

Although for some cuts the capacity from the "ON" peers alone is not sufficient, we observe that for other cuts, these capacities will actually be enough. In the following, we will divide all the cuts into three groups: The first group consists of those cuts whose capacity is so large that, the capacity from only the ON peers on the left side to only the ON peers on the right side is sufficient. The second group consists of the those cuts whose capacity is smaller such that, the capacity between only the ON peers is insufficient, but the capacity from only the ON peers on the right is still sufficient. The third group consists of all the other cuts with even smaller capacities. We can thus work with the sub-network of ON peers and use similar techniques as we did in the previous sections to analyze the first two groups of cuts, and consider the last group of cuts separately.

For any fixed destination peer t, let H_t be the sub-network that contains only the ON peer and peer t, which will be used to analyze groups 1 and 2 discussed above. Assume that there are Y of ON peers in the system other than peer t. Let D_m be the capacity of the cut W_m of the sub-network H_t , where there are m peers on one side, Y - mpeers on the other side. Construct another network \tilde{G} as follows: \tilde{G} has the same set of peers and the same set of edges as G. For each peer i in \tilde{G} , let its upload capacity \tilde{U}_i be μ if its corresponding upload capacity U_i in G is no less than μ , and let its upload capacity \tilde{U}_i be 0 otherwise, i.e., $\tilde{U}_i = \mu$ if $U_i \ge \mu$ and $\tilde{U}_i = 0$ if $U_i < \mu$. Then, the upload capacity U_i of each peer is given by the ON-OFF model with $\mathbf{P}(\tilde{U}_i = \mu) = p$ and $\mathbf{P}(\tilde{U}_i = 0) = 1 - p$. For any destination peer t, define \tilde{H}_t , \tilde{W}_m and \tilde{D}_m similarly as H_t , W_m and D_m . Obviously, \tilde{G} has smaller cut capacity than G. We have, for any m,

$$D_m \ge \tilde{D}_m$$

Thus, if the cut capacity \tilde{D}_m in the network \tilde{G} is sufficiently large, so is D_m . For a given Y, the expectation value of \tilde{D}_m is given by (similar to (4))

$$\mathbf{E}\left[\tilde{D}_{m}\right] = \begin{cases} \frac{u_{s}(Y-m)}{Y} + \frac{\mu}{N-1}m(Y-m+1) & \text{if } t \text{ is OFF}, \\ \frac{u_{s}(Y+1-m)}{Y+1} + \frac{\mu}{N-1}m(Y-m+1) & \text{if } t \text{ is ON}. \end{cases}$$

Since $\frac{Y-m}{Y} > \frac{Y+1-m}{Y+1}$, we have

$$\mathbf{E}\left[\tilde{D}_{m}\right] \geq \frac{u_{s}(Y+1-m)}{Y+1} + \frac{\mu}{N-1}m(Y-m+1).$$
(15)

Note that for any ϵ' , one can use the Chernoff bound to show that

$$\mathbf{P}(Y \ge (1 - \epsilon')p(N - 1)) \ge 1 - O(\exp(-\epsilon'^2 p^2(N - 1))).$$

In the following, we will focus on the case when the event $\mathcal{Y}_{\epsilon'} = \{Y = y, y \ge (1 - \epsilon')p(N - 1)\}$ holds. For the first group of cuts, we are interested in those cuts with $m < Y - \frac{2}{p}$. We have the following lemma.

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Lemma 13. For any $\epsilon \in (0,1)$, there exists N_0 such that if $N > N_0$ then for any $m < Y - \frac{2}{p}$, we have

$$\mathbf{P}\left(D_m \le (1-\epsilon)C_f | \mathcal{Y}_{\epsilon'}\right)$$
$$\le \mathbf{P}\left(\tilde{D}_m \le (1-\epsilon')\mathbf{E}[\tilde{D}_m | \mathcal{Y}_{\epsilon'}] | \mathcal{Y}_{\epsilon'}\right),$$

where $\epsilon' = 1 - \sqrt{1 - \epsilon}$.

Proof: If $m < \frac{2}{p}$, we have

$$\mathbf{E}[\tilde{D}_m|\mathcal{Y}_{\epsilon'}] \ge \left(1 - \frac{m}{Y+1}\right) u_s \ge \left(1 - \frac{m}{(1-\epsilon')p(N-1)+1}\right) u_s$$
$$\stackrel{N \to \infty}{\longrightarrow} u_s \ge C_f.$$

Therefore, for any $\epsilon' \in (0, 1)$, if N is large enough, we will have

$$\mathbf{E}[\tilde{D}_m | \mathcal{Y}_{\epsilon'}] \ge (1 - \epsilon')C_f.$$

Hence, for any $\epsilon \in (0,1)$, $\epsilon' = 1 - \sqrt{1-\epsilon}$, if N is large enough and $Y \ge (1-\epsilon')p(N-1)$, we have

$$\mathbf{P} \left(D_m \le (1 - \epsilon) C_f | \mathcal{Y}_{\epsilon'} \right)$$

$$\leq \mathbf{P} \left(D_m \le (1 - \epsilon') \mathbf{E} [\tilde{D}_m | \mathcal{Y}_{\epsilon'}] | \mathcal{Y}_{\epsilon'} \right)$$

$$\leq \mathbf{P} \left(\tilde{D}_m \le (1 - \epsilon') \mathbf{E} [\tilde{D}_m | \mathcal{Y}_{\epsilon'}] | \mathcal{Y}_{\epsilon'} \right).$$

Similarly, for any m such that $\frac{2}{p} \leq m \leq Y - \frac{2}{p},$ we will have

$$\begin{split} \mathbf{E}[\tilde{D}_m | \mathcal{Y}_{\epsilon'}] &\geq \frac{\mu}{N-1} m(Y-m+1) \qquad (\text{Using (15)}) \\ &\geq \frac{\frac{2}{p} \left((1-\epsilon')p(N-1) + 1 - \frac{2}{p} \right) \mu}{N-1} \\ &= \left(2(1-\epsilon') - \frac{4-2p}{p^2(N-1)} \right) \mu \xrightarrow{N \to \infty} 2(1-\epsilon')\mu, \end{split}$$

where the second inequality is due to the fact that the quadratic function m(Y - m + 1) of m reaches its minimum on $\left[\frac{2}{p}, Y - \frac{2}{p}\right]$ at $m = \frac{2}{p}$. Recall that

$$C_f \leq \frac{u_s}{N} + \mu \xrightarrow{N \to \infty} \mu.$$

Thus, for any $\epsilon' \in (0,1)$, if N is sufficiently large, then

$$\mathbf{E}[\tilde{D}_m | \mathcal{Y}_{\epsilon'}] \ge \sqrt{2}(1-\epsilon')\mu \ge (1-\epsilon')C_f.$$

We now have shown that for $m \leq Y - \frac{2}{p}$, the expected capacity of all the cuts \tilde{W}_m will approach to the optimal

streaming capacity C_f . That is, for $0 \le m \le Y - \frac{2}{p}$, if N is large enough, then

$$\mathbf{E}[\hat{D}_m | \mathcal{Y}_{\epsilon'}] \ge (1 - \epsilon')C_f,$$

and

$$\mathbf{P}\left(D_m \le (1-\epsilon)C_f | \mathcal{Y}_{\epsilon'}\right)$$
$$\le \mathbf{P}\left(\tilde{D}_m \le (1-\epsilon')\mathbf{E}[\tilde{D}_m | \mathcal{Y}_{\epsilon'}] | \mathcal{Y}_{\epsilon'}\right).$$

For $m \ge Y - \frac{2}{p}$, unfortunately the total upload capacities between the ON peers alone is insufficient. Luckily, for the second group of cuts (to be defined shortly), if we consider the capacities from the ON peers on the left to all peers on the right not only the ON peers on the right, we may still receive adequate capacity. Consider a cut in the whole network. Let k be the number of OFF peers on the left. Denote $D_{m,k}$ and $\tilde{D}_{m,k}$ as the cut capacity from the ON peers on the left to all the peers on the right in G and \tilde{G} , respectively. For the second group of cuts, we are interested in those cuts with $m \ge Y - \frac{2}{p}$ and $N - k - m > \frac{2}{p^2}$. The following result holds.

Lemma 14. For any $\epsilon \in (0,1)$, there exists N_0 such that, if $N > N_0$, then for any m and k such that $m \ge Y - \frac{2}{p}$ and $N - k - m > \frac{2}{p^2}$, we have

$$\mathbf{P}\left(D_{m,k} \leq (1-\epsilon)C_f | \mathcal{Y}_{\epsilon'}\right)$$
$$\leq \mathbf{P}\left(\tilde{D}_{m,k} \leq (1-\epsilon')\mathbf{E}[\tilde{D}_m | \mathcal{Y}_{\epsilon'}] | \mathcal{Y}_{\epsilon'}\right),$$

where $\epsilon' = 1 - \sqrt{1 - \epsilon}$.

Proof: Similarly to (15), the expectation of $\tilde{D}_{m,k}$ would be bounded as

$$\mathbf{E}\left[\tilde{D}_{m,k}|\mathcal{Y}_{\epsilon'}\right] \geq \frac{Y-m+1}{Y+1}u_s + \frac{m(N-k-m)}{N-1}\mu.$$

Note that $m \ge Y - \frac{2}{p}$ and $N - k - m > \frac{2}{p}$, therefore,

$$\mathbf{E}\left[\tilde{D}_{m,k}|\mathcal{Y}_{\epsilon'}\right] \geq \frac{(Y-\frac{2}{p})\frac{2}{p}}{N-1}\mu$$
$$\geq \frac{2(1-\epsilon')(N-1)-\frac{4}{p^2}}{N-1}\mu$$
$$\stackrel{N\to\infty}{\longrightarrow} 2(1-\epsilon')\mu.$$

Recall that

$$C_f \leq \frac{u_s}{N} + \mu \xrightarrow{N \to \infty} \mu.$$

As a result, for any $\epsilon' \in (0,1)$, when $\mathcal{Y}_{\epsilon'}$ holds, we have for sufficiently large N,

$$\mathbf{E}\left[\tilde{D}_{m,k}\right] \ge (1-\epsilon')C_f,$$

and

$$\mathbf{P}\left(D_{m,k} \leq (1-\epsilon)C_f | \mathcal{Y}_{\epsilon'}\right)$$
$$\leq \mathbf{P}\left(\tilde{D}_{m,k} \leq (1-\epsilon')\mathbf{E}[\tilde{D}_{m,k} | \mathcal{Y}_{\epsilon'}] | \mathcal{Y}_{\epsilon'}\right).$$

Lemma 13 and Lemma 14 allow us to use a network \tilde{G} with ON-OFF upload capacity to bound the performance of the original network when $m < Y - \frac{2}{p}$ or $m \ge Y - \frac{2}{p}$ and $N - k - m + 1 > \frac{2}{p}$. We can then apply our previous results to analyze the network \tilde{G} and obtain an upper bound on probability that the capacity of cuts in the original network is smaller than $(1 - \epsilon)C_f$. The detailed proof can be found in the proof of Theorem 12 in Appendix H.

Finally, for the third group of cuts, we are interested in those cuts with $m \ge Y - \frac{2}{p}$ and $N - k - m < \frac{2}{p}$. For these cuts, the capacities from the ON peers alone will be insufficient. We need to take account all the peer capacities in the network. Under such situation, we will not need to differentiate the ON peers and OFF peers. Let us consider a cut $V_{\tilde{k}}$ where there are \tilde{k} peers on the left and $N - \tilde{k}$ peers on the right. The condition $N - k - m < \frac{2}{p^2}$ will be equivalent to $N - \tilde{k} < \frac{2}{p^2}$ when we do not care about the actual number of ON peers. The following bound holds for this case.

Lemma 15. When $N - \tilde{k} < \frac{2}{p^2}$, for any $\epsilon \in (0, 1)$, if N is sufficiently large, we have

$$\mathbf{P}(C_{\tilde{k}} \le (1-\epsilon)\mathbf{E}[C_{\tilde{k}}]) \le e^{-\tilde{p}\tilde{k}\left\lfloor (N-\tilde{k})q\frac{\epsilon^2}{2} + O(q^2) \right\rfloor},$$

where $\tilde{p} = \frac{\mu}{U_{\text{max}}}$ and $q = \frac{M}{N-1}$.

The proof of 15 is in Appendix G. Until now, we have divided all the cuts into three different groups: 1)when the capacities between ON peers are sufficient, 2)when the capacities from the ON peers to all kinds of peers are adequate and 3)when the capacities from the ON peers alone are insufficient. For the first two groups of cuts, since the capacities from the ON peers alone is sufficient, we can apply a similar technique as we did for the ON-OFF model to show that all the cuts will have a capacity lager than $(1 - \epsilon)C_f$ with high probability. For the last group of cuts, Lemma 15 provides us an upper bound on the probability that the capacity of any cut in this group is smaller than $(1 - \epsilon)C_f$. By treating these three groups of cuts separately, we can then prove Theorem 12. The detailed proof can be found in Appendix H.

V. SIMULATION

In this section, we provide simulation results to verify our analytical results in previous sections. We simulate a P2P network with N = 10000 peers and one server. Although the analytical results in this paper focus on the ON-

OFF model for peers' upload capacity, here we provide simulation results both for the ON-OFF model and a uniform distribution model. In the ON-OFF model, each user has an ON probability of p. When a user is ON, it contributes an upload capacity u = 10. On the other hand, in the uniform distribution model, the upload capacity of each peer is uniformly distributed between [0, 20]. Further, each peer chooses the same number of downstream neighbors and divides its upload capacity evenly among these neighbors, regardless of its upload capacity. In both cases, the server has a capacity of $u_s = 20$. The optimal streaming capacity is thus $C_f = 9.002$ for the ON-OFF model with p = 0.9, and $C_f = 5.002$ for the ON-OFF model with p = 0.5 and for the uniform distribution model. We vary the number of downstream neighbors of each user from $10 \log N = 90$ to $80 \log N = 720$, which correspond to 0.9%and 7.2% of the total number of peers N. For each choice of the number of downstream neighbors, we generate the network for 200 times. During each iteration, all users select their downstream neighbors randomly as described in section II-B, and we use the algorithm in [28] (a modified push-relabel algorithm) to find the min-min cut from the source to all the users and compare it with $(1 - \epsilon)C_f$. We count the number of times that the min-min cut of the network is larger than $(1-\epsilon)C_f$ and plot the probability for that to happen as the number of downstream neighbors of each peer varies. The result is shown in Fig. 2, where we simulate four different combinations of p(for the ON-OFF model) and ϵ . We can observe that, using pure random selection, when p = 0.5 for the ON-OFF model and when the number of downstream neighbors of each peer is more than $40 \log N = 360$ (3.6% of N), the success probability that the system could sustain a streaming rate higher than 70% of the optimal streaming capacity is greater than 0.9. If p = 0.9 for the ON-OFF model, the number of downstream neighbors needed by each peer to achieve the same success probability of 0.9 reduces to $30 \log N = 270$ (2.7% of N). Further, we can observe that with the same ON probability, when we increase ϵ , the required number of downstream neighbors to achieve the same success probability of 0.9 decreases. These observations verify our remarks following Theorem 1 that M needs to be larger if ϵ is smaller or p is smaller. We also observe that, when the upload capacity of each peer follows the uniform distribution and when the number of downstream neighbors of each peer is more than $40 \log N = 360$ (3.6% of N), the success probability of sustaining more than 70% of the optimal streaming capacity is almost 1. This suggests that our analytical result is still valid for other models of peer upload capacity.

We note that in the above simulation results, the number of neighbors required to achieve a high success probability is still quite large for a network with 10000 peers. Although our analytical results show that having $\Omega(\log N)$ neighbors is sufficient to achieve close-to-optimal streaming capacity with high probability when $N \to \infty$, the actual number of neighbors required depends on the constant factor before the $\log N$ term. As these simulations show, while the random peer-selection and uniform rate-allocation algorithm is the easiest to implement and the most robust to changes, it does suffer some performance penalty in terms of the number of neighbors required. The hybrid algorithm proposed in Section III is designed to further improve the constant factor. We then run the hybrid algorithm on a P2P network with N = 10000 peers with ON probability p = 0.5. We first choose the parameter θ to be 1/3 (i.e., each user allocates 1/3 of its upload capacity uniformly among its downstream neighbors and performs the more sophisticated rate-allocation with the remaining 2/3 upload capacity as described in Section III). The result is shown in Fig 3. We first notice that the same general trend still holds for the relationship between the



Number of downstream neighbors M as a fraction of N

Fig. 2. The success probability versus the number of downstream neighbors under uniform rate-allocation

ON probability p, the approximation ratio ϵ and the success probability, as discussed earlier for the uniform rateallocation scheme. However, the number of neighbors required is reduced by an order of magnitude. For example, in Fig 3 when the number of downstream neighbors of each peer is more than $4 \log N = 36$ (0.36% of N), the probability that the system can sustain a streaming rate higher than 80% of the optimal streaming capacity is already almost 1. In contrast, we observe from Fig 2 that if we use uniform rate allocation, each peer needs more than $90 \log N = 810$ (8.1% of N) downstream neighbors to achieve the same performance. Hence, the hybrid algorithm reduces the required number of downstream neighbors of each peer by more than 20 times, while still retaining the simplicity and robustness of the random peer-selection scheme. In addition, we simulate the hybrid algorithm with $\theta = 2/3$ and $\theta = 1$. As we can see from Fig. 2, the performance of the hybrid algorithm with $\theta = 2/3$ is almost identical to that of $\theta = 1/3$. On the other hand, by comparing the two curves with $\theta = 1/3$ and $\theta = 1$ but same p and ϵ , we can observe that the hybrid algorithm with $\theta = 1/3$ has a higher success probability than the algorithm. Therefore, this simulation result confirms our argument in Section III that pure "sophisticated" rate-allocation algorithm may sacrifice the capacity of cuts other than the last-hop cuts.

We next simulate the performance of both the uniform and the hybrid rate-allocation algorithm when the total number of users N changes. We vary the total number of users in the systems from N = 100 to N = 6400. The results are shown in Fig. 4 and Fig. 5. For the results of the uniform rate-allocation algorithm in Fig. 4, we choose the parameters p = 0.9 and $\epsilon = 0.2$. Each curve corresponds to a different choice of M from $M = 30 \log N$ to $M = 80 \log N$. An interesting observation is that when M is small (e.g. $M = 30 \log N$), the performance in fact degrades as N increases. The reason is that when N is small, M may be even larger than N, in which case we



Fig. 3. The success probability versus the number of downstream neighbors under hybrid rate-allocation

use M = N and the network becomes fully connected. However, as N increases, the sparse connectivity and the negative effect of low M will eventually kick in. On the other hand, when M is sufficiently large $(M = 80 \log N)$, the success probabilities under all different values of N are always 1. For the results of the hybrid rate-allocation algorithm in Fig. 5, we choose the parameters p = 0.5 and $\epsilon = 0.2$. Each curve corresponds to a different choice of M from $M = 2 \log N$ to $M = 5 \log N$. We observe that the performance of the hybrid rate-allocation algorithm is less sensitive to the total number of users N. Under the same value of M, the success probability remains on the same level as N varies. On the other hand, we can still see that when M is sufficiently large, the success probability becomes 1 for all different values of N.

VI. CONCLUSION

In this paper, we study the streaming capacity of sparsely-connected P2P networks. We show that even with a random peer-selection algorithm and uniform rate allocation, as long as each peer maintains $\Omega(\log N)$ downstream neighbors, the system can achieve close-to-optimal streaming capacity with high probability when the network size is large. These results provide important new insights on the streaming capacity of large P2P network with a sparse topology. One such insight is that the capacity of the last cut (i.e., the capacity from direct upstream neighbors) is often the bottleneck. We then use this insight to improve the peer-selection and rate-allocation algorithm to further optimize the achievable streaming capacity. Specifically, we design a hybrid algorithm that uses a slightly more sophisticate rate-allocation algorithm to improve the capacity and to reduce the constant factor in the $\Omega(\log N)$ result. This new algorithm still retains the simplicity and robustness of the random peer-selection scheme, but it



Fig. 4. The success probability versus N under uniform rate-allocation



Fig. 5. The success probability versus N under hybrid rate-allocation

significantly reduces the number of neighbors required to achieve a certain performance guarantee.

Throughout this paper, we have assumed a uniformly-random peer-selection scheme. It is highly likely that more sophisticated peer-selection schemes (albeit with a higher complexity) may lead to even better performance, e.g., an even smaller factor α . For instance, one may assign a larger number of downstream neighbors to a peer with a larger upload capacity. However, we caution that the resulting performance improvement is not automatic. As we have seen

in Section III for the hybrid algorithm, the effect of *local* improvement on the *global* performance can be difficult to quantify. Thus, the insights obtained from our analysis may be used to guide the design of more sophisticated algorithms. Further, our analysis has focused on attaining a close-to-optimal streaming rate. In practice, it may be important to estimate the value of the optimal streaming rate. It is possible to use the structure of the hybrid algorithm (e.g., Equation (10)) to design a distributed algorithm that can determine an estimate of this streaming rate. Finally, this paper has focused on P2P live-streaming systems. For future work, we will investigate whether similar insights can also be extended to P2P video-on-demand services, which have also become increasingly popular.

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APPENDIX A

PROOF OF LEMMA 9

Proof: For the "only if" part, suppose that for some subset S of V, (12) does not hold. Note that all peers in S can only receive capacity from the peers in U(S). Then, the total upload capacity of the peers in U(S) could not satisfy the total demand of the peers in S. Hence, there must be some peer in S that could not achieve the required streaming rate, which leads to a contradiction. Therefore, (12) must hold for any subset S of V if there exists a feasible assignment for (9).

To prove the "if" part, we will use the max-flow min-cut theorem. Construct a new network G_F with two vertexsets V_1 and V_2 , and two additional vertices s and t. Each peer i has a copy in both V_1 and V_2 . Add a directed edge from node i in V_1 to node j in V_2 with capacity u_i if i is a upstream neighbor of j. Further, add a directed edge from s to every node i in V_1 with capacity u_i , and a directed edge from every node j in V_2 to t with capacity R_j . For each set of feasible values of C_{ij} 's in (9), one can construct a corresponds feasible flow in this new flow network, where the value of C_{ij} equals to the amount of the flow from node i in V_1 to node j in V_2 .

We claim that there exists a feasible assignment of C_{ij} 's such that condition (9) holds if and only if the maximum flow from s to t in the above network G_F is equal to $\sum_{j \in V} R_j$. To see this, note that condition (9) implies that every peer could receive an allocated capacity of at least R_j from its upstream neighbors, while the total capacity allocated from each peer i to its downstream neighbors is no more than u_i . Then, by further reducing C_{ij} 's appropriately, we can find a set of C_{ij} 's such that every peer receives an allocated capacity equal to R_j . By letting C_{ij} be the flow in G_F from node i in V_1 to node j in V_2 , we obtained a flow from s to t, where the amount of outgoing flow from every node i in V_1 is no greater than u_i and the amount of flow that comes into every node j in V_2 is exactly equal to R_j . Clearly, one can then construct a flow from s to t with value equal to $\sum_{j \in V} R_j$. On the other hand, if there exists a flow from s to t with value equal to $\sum_{j \in V} R_j$. In addition, the outgoing flow from every node i in V_1 must be no greater than its incoming flow, which is at most u_i . Therefore, every peer j receives an allocated upload capacity from its upstream neighbors equal to R_j , while the total capacity allocated from every node i in V_1 must be no greater than its incoming flow, which is at most u_i . Therefore, every peer j



Fig. 6. Minimum cut of G_F

Now to prove the "if" part of the statement of Lemma 9, we only need to show that if (12) holds for any subset $S \subset V$, then the maximum flow of the network is equal to $\sum_{j \in V} R_j$. To show this part, assume in the contrary that (12) holds for any subset $S \subset V$ but the maximum flow of the network is less than $\sum_{j \in V} R_j$. By the max-flow min-cut theorem, this assumption means that the minimum cut of the network is less than $\sum_{j \in V} R_j$. Therefore, there exists a cut (X_1, X_2) with capacity less than $\sum_{j \in V} R_j$, where $s \in X_1$ and $t \in X_2$. Let $T_2 = X_2 \cap V_2$. Let $\mathcal{U}'(T_2)$ denote the set of vertices in V_1 that have an edge to at least one vertex in T_2 . Next, construct another cut (X'_1, X'_2) by moving all the nodes in $\mathcal{U}'(T_2)$ to X_2 . The significant of this new cut is that all the vertices in $\mathcal{U}'(T_2)$ are in X'_2 , which will be used to construct a contradiction. First, we claim that the capacity of the new cut $C(X'_1, X'_2)$ is no greater than the capacity of the old cut (X_1, X_2) . To see this, consider a vertex v_i in $\mathcal{U}(T_2)$ but not in X_2 . If we move v_i from X_1 to X_2 , we will get a new cut (X''_1, X''_2) . Then the edge (s, v_i) crosses the new cut (X''_1, X''_2) , but does not cross the old cut (X''_1, X''_2) . Further, all other edges cross either both cuts or no cut (see Fig 6 for illustration). As a result,

$$C(X_1'', X_2'') = C(X_1, X_2) + C(s, v_i) - C(v_i, T_2)$$

Since v_i is the upsteam neighbor of at least one node in T_2 , we have $C(v_i, T_2) \ge u_i$. In addition, note that $C(s, v_i) = u_i$. We thus have $C(X_1'', X_2'') \le C(X_1, X_2)$. Hence, the capacity of the new cut $C(X_1'', X_2'')$ is no greater than the capacity of the old cut $C(X_1, X_2)$, which is less than $\sum_{j \in V} R_j$. Therefore, as we move more peer in $\mathcal{U}'(T_2)$ from X_1 to the other side of the cut, the capacity of the cut cannot increase. We thus have

 $C(X'_1, X'_2) \leq C(X''_1, X''_2) < \sum_{j \in V} R_j$. Now let $T_1 = X'_2 \cap V_1$. We have $\mathcal{U}(T_2) \subseteq T_1$. Note that the edges from s to T_1 and the edges from $V_2 \setminus T_2$ to t cross the cut (X'_1, X'_2) . Thus, the capacity of the cut is at least $\sum_{i \in T_1} u_i + \sum_{j \in V_2 \setminus T_2} R_j$, i.e.,

$$\sum_{i\in T_1} u_i + \sum_{j\in V_2\backslash T_2} R_j \leq C(X_1', X_2') < \sum_{j\in V} R_j.$$

Note that there is a one-to-one mapping between vertices in V_2 and the nodes in V. Therefore, $\sum_{j \in V} R_j = \sum_{j \in V_2} R_j$. Since $\mathcal{U}(T_2) \subseteq T_1$, we then have,

$$\sum_{j \in T_2} R_j > \sum_{i \in T_1} u_i \ge \sum_{i \in \mathcal{U}(T_2)} u_i,$$

which contradicts with (12). The result of the lemma thus follows.

APPENDIX B

PROOF OF PROPOSITION 5

Proof: We can write $\sum_{j \in \tilde{W}_2} C_{ij} = L_i \cdot \frac{u}{M}$, where L_i is the number of downstream neighbors of peer *i* in \tilde{W}_2 . As mentioned above, peer *i* select *M* downstream neighbors from \tilde{N} different peers. Consider all the potential downstream neighbors $j \in \tilde{W}_2$. Let I_{ij} be the indicator function of the event that peer *j* is a downstream neighbors of *i*. Clearly, I_{ij} has a Bernoulli distribution with parameter M/\tilde{N} . Moreover, the number of downstream neighbors in \tilde{W}_2 would be equal to the summation of all the I_{ij} 's over *j*, i.e., $L_i = \sum_{j \in \tilde{W}_2} I_{ij}$ and follows a hyper-geometric distribution. According to Theorem 4 in [25], if \tilde{I}_{ij} , $j \in \tilde{W}_2$ are i.i.d. Bernoulli random variables such that \tilde{I}_{ij} has the same marginal distribution as I_{ij} , we will have, for any real $t \mathbf{E} \left[e^{t \sum_{j \in \tilde{W}_2} I_{ij}} \right] \leq \mathbf{E} \left[e^{t \sum_{j \in \tilde{W}_2} \tilde{I}_{ij}} \right]$. This means that we could use the moment generating function of a binomial random variable, which is the summation of i.i.d. Bernoulli random variables, to bound the moment generating function of the hyper-geometric random variable. Letting $t = -\theta$, we then have, for each $i \in \tilde{W}_1$

$$\mathbf{E}\left[e^{-\theta \frac{u}{M}\sum_{j\in\bar{W}_{2}}I_{ij}}\right] \leq \mathbf{E}\left[e^{-\theta \frac{u}{M}\sum_{j\in\bar{W}_{2}}\tilde{I}_{ij}}\right]$$
$$=\left(\mathbf{E}\left[e^{-\theta \frac{u}{M}\tilde{I}_{ij}}\right]\right)^{r} = 1 - \frac{M}{\tilde{N}}\left(1 - e^{-\theta \frac{u}{M}}\right)$$
(16)

Note that, $1 - \frac{M}{\tilde{N}} \left(1 - e^{-\theta \frac{u}{M}}\right) \le \exp\left[\frac{M}{\tilde{N}} \left(e^{-\theta \frac{u}{M}} - 1\right)\right]$, since $0 \le \frac{M}{\tilde{N}} \left(1 - e^{-\theta \frac{u}{M}}\right) \le 1$, and $1 - x \le e^{-x}$ when $0 \le x \le 1$. Therefore, substituting the above inequality into (16) yields

$$\mathbf{E}\left[e^{-\theta \frac{u}{M}\sum_{j\in \tilde{W}_2}I_{ij}}\right] \le \exp\left[r\frac{M}{\tilde{N}}\left(e^{-\theta \frac{u}{M}}-1\right)\right].$$

For different peers in \tilde{W}_1 , they will select their downstream neighbors independently. Hence, $\sum_{j \in \tilde{W}_2} C_{ij}$ are independent across *i*. Therefore,

$$\mathbf{E}\left[e^{-\theta\sum_{i\in\tilde{W}_{1}}\sum_{j\in\tilde{W}_{2}}C_{ij}}\right] = \left(\mathbf{E}\left[e^{-\theta\sum_{j\in\tilde{W}_{2}}C_{ij}}\right]\right)^{q}$$
$$= \left(\mathbf{E}\left[e^{-\theta\frac{u}{M}\sum_{j\in\tilde{W}_{2}}I_{ij}}\right]\right)^{q} \le \exp\left[\frac{Mrq}{\tilde{N}}\left(e^{-\theta\frac{u}{M}}-1\right)\right].$$

APPENDIX C

PROOF OF LEMMA 6

Proof: By Chernoff bounds, we have for $\theta > 0$

$$\mathbf{P}(D_m \le (1-\epsilon)\mathbf{E}[D_m]|Y = y, t \text{ is ON})$$

$$\le \frac{\mathbf{E}\left[e^{-\theta D_m}|Y = y\right]}{e^{-(1-\epsilon)\theta\mathbf{E}[D_m|Y = y]}} = e^{\phi(\theta) + \phi_s(\theta)},$$
(17)

where

$$\phi(\theta) = \log \mathbf{E} \left[e^{-\theta \sum_{j=1}^{m} \sum_{i=m+1}^{y+1} C_{ji}} \right] + \theta(1-\epsilon)m(y-m+1)\frac{u}{N-1}; \phi_s(\theta) = \log \mathbf{E} \left[e^{-\theta \sum_{i=m+1}^{y+1} C_{si}} \right] + \theta(1-\epsilon)(y-m+1)\frac{u_s}{y+1}.$$

Now we apply Proposition 5. Recall that we define a cut on H_t by dividing peers into sets W_m and W_m^c . We could also view W_m and W_m^c as subsets of some cut V_k and V_k^c of network G. We need to exclude the server from W_m since it has a different upload capacity. For each peer in $W_m \setminus s$, it will choose M downstream neighbors randomly from the entire network. Hence, $\tilde{V} = V$. According to proposition 5, we have $q = |W_m \setminus s| = m$, $r = |W_m^c| = y - m + 1$ and $|\tilde{V}| = N$. Therefore, using (5), we have,

$$\begin{split} \phi(\theta) \leq &\log\left\{\exp\left[Mm\frac{y+1-m}{N-1}\left(e^{-\theta\frac{u}{M}}-1\right)\right]\right\} \\ &+ \theta(1-\epsilon)(y+1-m)\frac{u}{N-1} \\ = &\frac{1}{N-1}\left[Mm\left(e^{-\theta\frac{u}{M}}-1\right)+\theta(1-\epsilon)u\right](y+1-m). \end{split}$$

Note that the server only choose neighbors from the y + 1 ON peers, $|\tilde{V}| = y + 1$. Using similar techniques, for the server, we can bound $\phi_s(\theta)$ by

$$\phi_s(\theta) \le \frac{1}{y+1} \left[M \left(e^{-\theta \frac{u_s}{M}} - 1 \right) + \theta (1-\epsilon) u_s \right] (y+1-m).$$

Define

$$\tilde{\phi}(\theta) \triangleq M\left(e^{-\theta \frac{u}{M}} - 1\right) + \theta(1 - \epsilon)u;$$
$$\tilde{\phi}_s(\theta) \triangleq M\left(e^{-\theta \frac{u_s}{M}} - 1\right) + \theta(1 - \epsilon)u_s$$

The $\phi(\cdot)$ and $\phi_s(\cdot)$ can be written as

$$\phi(\theta) \le \frac{1}{N-1} \tilde{\phi}(\theta) m(y+1-m);$$

$$\phi_s(\theta) \le \frac{1}{y+1} \tilde{\phi}_s(\theta) (y+1-m).$$

Let $\tilde{\phi}_{\min}$ and $\tilde{\phi}_{s,\min}$ be the minimum of $\tilde{\phi}(\theta)$ and $\tilde{\phi}_s(\theta)$ respectively, over $\theta > 0$. It is easy to see $\tilde{\phi}_{\min} = \tilde{\phi}_{s,\min} < 0$. Also since $\tilde{\phi}$ and $\tilde{\phi}_s$ is convex on $\theta > 0$, these minimum is attainable. Let θ_{\min} and $\theta_{s,\min}$ be the minimizer respectively. We must have

$$\tilde{\phi}_s(\theta_{s,\min}) = \tilde{\phi}_{s,\min} = \tilde{\phi}_{\min} \le \tilde{\phi}(\theta_{s,\min}).$$
(18)

One can show that $\theta_{s,\min} = -\frac{M}{u_s} \log(1-\epsilon)$. Note that for 0 < a < 1 and $0 \le x \le 1$, we have $(1-x)^a \le 1-ax$ since $(1-x)^a$ is concave and its derivative at 0 is -a. Moreover, for $0 \le x \le 1$, one can see that $(1-x) \log(1-x) \ge x^2/2 - x$ by checking $\frac{d}{dx}(1-x) \log(1-x) - (x^2/2 - x) = -\log(1-x) - x \ge 0$ and $(1-x) \log(1-x) = x^2/2 - x$ when x = 0. Then, substituting $\theta_{s,\min}$ into (18) and using the above relationship, we have

$$\begin{split} \bar{\phi}_s(\theta_{s,\min}) &\leq \bar{\phi}(\theta_{s,\min}) \\ = & M \left[(1-\epsilon)^{\frac{u}{u_s}} - 1 \right] - M \frac{u}{u_s} (1-\epsilon) \log(1-\epsilon) \\ &\leq & M \left[1 - \frac{u}{u_s} \epsilon - 1 - \frac{u}{u_s} \left(\frac{\epsilon^2}{2} - \epsilon \right) \right] = -M \frac{u}{u_s} \frac{\epsilon^2}{2}. \end{split}$$

Consequently,

$$\begin{split} & m\phi(\theta_{s,\min}) + \phi_s(\theta_{s,\min}) \\ \leq & \tilde{\phi}(\theta_{s,\min})m(y+1-m)/(N-1) \\ & + & \tilde{\phi}_s(\theta_{s,\min})(y+1-m)/(y+1) \\ \leq & - \left(m\frac{y+1-m}{N-1} + \frac{y+1-m}{y+1}\right)M\frac{u}{u_s}\frac{\epsilon^2}{2} \end{split}$$

Since (17) holds for any $\theta > 0$, letting $\theta = \theta_{s,\min}$ yields

$$\begin{split} &\mathbf{P}(D_m \leq (1-\epsilon)\mathbf{E}[D_m]|Y = y, t \text{ is ON}) \\ &\leq \exp(m\phi(\theta_{s,\min}) + \phi_s(\theta_{s,\min})) \\ &\leq \exp\left[-\left(Mm\frac{y+1-m}{N-1} + M\frac{y+1-m}{y+1}\right)\frac{u}{u_s}\frac{\epsilon^2}{2}\right]. \end{split}$$

Similarly, one can show that if t is OFF, we have

$$\mathbf{P}(D_m \le (1-\epsilon)\mathbf{E}[D_m]|Y = y, t \text{ is OFF})$$

$$\le \exp\left[-\left(Mm\frac{y+1-m}{N-1} + M\frac{y-m}{y}\right)\frac{u}{u_s}\frac{\epsilon^2}{2}\right].$$

Since $\frac{y+1-m}{y+1} \ge \frac{y-m}{y}$, we have

$$\mathbf{P}(D_m \le (1 - \epsilon)\mathbf{E}[D_m]|Y = y, t \text{ is ON})$$
$$\le \mathbf{P}(D_m \le (1 - \epsilon)\mathbf{E}[D_m]|Y = y, t \text{ is OFF})$$

Hence,

$$\mathbf{P}(D_m \le (1-\epsilon)\mathbf{E}[D_m]|Y=y)$$

$$\le \mathbf{P}(D_m \le (1-\epsilon)\mathbf{E}[D_m]|Y=y, t \text{ is OFF})$$

$$\le \exp\left[-\left(Mm\frac{y+1-m}{N-1} + M\frac{y-m}{y}\right)\frac{u}{u_s}\frac{\epsilon^2}{2}\right].$$

APPENDIX D

PROOF OF LEMMA 7

Proof: For $\gamma = \eta p$. We then have

$$\mathbf{P}\left(\bigcup_{m=0}^{Y-1}\tilde{\mathcal{B}}_{m}\right) \leq \sum_{y=0}^{\lceil \gamma N \rceil - 1} {\binom{N-1}{y-1}} p^{y} (1-p)^{N-1-y} \mathbf{P}\left(\bigcup_{m=0}^{y} \tilde{\mathcal{B}}_{m} \middle| Y = y\right) + \sum_{y=\lceil \gamma N \rceil}^{N-1} {\binom{N-1}{y-1}} p^{y} (1-p)^{N-1-y} \mathbf{P}\left(\bigcup_{m=0}^{y} \tilde{\mathcal{B}}_{m} \middle| Y = y\right).$$

The first term satisfies,

$$\sum_{y=0}^{\lceil \gamma N \rceil - 1} {\binom{N-1}{y-1}} p^y (1-p)^{N-1-y} \mathbf{P} \left(\bigcup_{m=0}^y \tilde{\mathcal{B}}_m \middle| Y = y \right)$$

$$\leq \mathbf{P} (Y < \lceil \gamma N \rceil - 1) \leq e^{-2 \frac{(p(N-1) - (\lceil \gamma N \rceil - 1))^2}{p(N-1)}}$$

$$= O(\exp(-(1-\eta)^2 pN)),$$

where the last inequality follows from the Chernoff bound. For m = 0, $D_m = u_s \ge C_f$. Therefore the probability $\mathbf{P}(\tilde{\mathcal{B}}_0|Y=y)$ is always 0. We can then take the summation from m = 1. We have,

$$\mathbf{P}\left(\bigcup_{m=0}^{y-1} \tilde{\mathcal{B}}_{m} \middle| Y = y\right) \leq \sum_{m=1}^{y-1} {y \choose m} \beta^{m \frac{y-m+1}{N-1} + \frac{y-m}{y}} \\
\leq \beta^{\gamma} \sum_{m=1}^{y-1} {y \choose m} \beta^{m\gamma \frac{y-m}{y}} \\
= \beta^{\gamma} \left[\sum_{m=1}^{\lfloor y/2 \rfloor} {y \choose m} \beta^{m\gamma \frac{y-m}{y}} + \sum_{m=\lfloor y/2 \rfloor+1}^{y-1} {y \choose m} \beta^{m\gamma \frac{y-m}{y}}\right] \\
\leq \beta^{\gamma} \left[\sum_{m=1}^{\lfloor y/2 \rfloor} {y \choose m} \beta^{\frac{m\gamma}{2}} + \sum_{m=\lfloor y/2 \rfloor+1}^{y-1} {y \choose m} \beta^{\frac{(y-m)\gamma}{2}}\right] \\
= 2\beta^{\gamma} \left[\left(1 + \beta^{\frac{\gamma}{2}}\right)^{y} - 1 \right].$$
(19)

We then have

$$\begin{split} &\sum_{y=\lceil\gamma N\rceil}^{N-1} \binom{N-1}{y} p^y (1-p)^{N-1-y} \mathbf{P} \left(\bigcup_{m=0}^{y-1} \tilde{\mathcal{B}}_m \middle| Y = y \right) \\ &\leq 2 \sum_{y=\lceil\gamma N\rceil}^{N-1} \binom{N-1}{y} p^y (1-p)^{N-1-y} \beta^{\gamma} \left[\left(1 + \beta^{\frac{\gamma}{2}} \right)^y - 1 \right] \\ &\leq \sum_{y=0}^{N-1} \binom{N-1}{y} 2 \beta^{\gamma} \left(p(1+\beta^{\frac{\gamma}{2}}) \right)^y (1-p)^{N-1-y} \\ &= 2\beta^{\gamma} \left[\left(1 + p\beta^{\frac{\gamma}{2}} \right)^{N-1} - 1 \right]. \end{split}$$

Then, plugging in the value of β will yields (6). For m = y,

$$\mathbf{P}\left(\left.\tilde{\mathcal{B}}_{y}\right|Y=y\right) = \mathbf{P}\left(\left.D_{y}\leq(1-\epsilon)C_{f}\right|Y=y\right)$$
$$\leq e^{-\left(My\frac{1}{N-1}\right)\frac{u}{u_{s}}\frac{\epsilon'^{2}}{2}} = \beta^{\gamma}$$

(7) then follows trivially.

APPENDIX E

PROOF OF LEMMA 11

Proof: Let Y be the number of ON users in the system, which is a random variable with binomial distribution Bin(N, p). For any subset S of V, define T_S as the number of ON peers that are the upstream neighbors of at least one peer in S, i.e. $T_S = |\{i \in \mathcal{U}(S) | i \text{ is ON}\}|$. Let S = |S| be the number of peers in S. Then, $\sum_{i \in \mathcal{U}(S)} u_i = uT_S$,

and the following two events are equivalent (defined as Γ_{S}):

$$\Gamma_{\mathcal{S}} \triangleq \left\{ \theta \sum_{i \in \mathcal{U}(\mathcal{S})} u_i \leq \sum_{j \in \mathcal{S}} \left[(1 - \epsilon) C_f - \sum_{i \in V} C_{ij}^U \right] \right\}$$
$$= \left\{ \theta u T_{\mathcal{S}} + \sum_{i \in V} \sum_{j \in \mathcal{S}} C_{ij}^U \leq S(1 - \epsilon) C_f \right\}.$$
(20)

In the last event of (20), the first term of the left hand side is the capacity from the more sophisticated allocation and the second term is the capacity from uniform allocation. We divide the proof into two parts according to the value of S.

1) We first consider the case when S is small, i.e., $S \leq \delta N$, where $\delta \in (0, \theta/2)$ is a small constant that does not depend on N. We will show that, when s is very small, the capacity of the more sophisticated allocation $\theta u T_S$ alone will be sufficient with high probability, i.e., it may be larger than $S(1 - \epsilon)C_f$. Recall that $C_f =$ $\min \{up + \frac{u_s}{N}, u_s\} \leq u (p + \frac{u_s}{uN})$. Let $p' = p + \frac{u_s}{uN}$. Then, for any $\epsilon > 0$, there exists N_0 such that whenever $N > N_0, p' . We thus have <math>\theta u T_S < S(1 - \epsilon)C_f$ implies $T_S < (1 - \epsilon)Sp'/\theta$. Therefore,

$$\mathbf{P}(\Gamma_{\mathcal{S}}) \leq \mathbf{P}\left(\theta u T_{\mathcal{S}} \leq S(1-\epsilon)C_{f}\right)$$

$$\leq \mathbf{P}\left(T_{\mathcal{S}} < (1-\epsilon)Sp'/\theta\right).$$
(21)

Next, we are going to show that the probability that $T_{\mathcal{S}} < (1-\epsilon)Sp'/\theta$ for some $\mathcal{S} \subset V$ is very small. To prove this, we first make the following claim: if there exists a set of peers \mathcal{S} such that $T_{\mathcal{S}} < (1-\epsilon)Sp'/\theta$, then there exists another set of peers \mathcal{S}' such that

$$T_{\mathcal{S}'} \in \mathcal{I}_{\mathcal{S}'}(\epsilon, p') \triangleq \left[\frac{(1-\epsilon)(S'-1)p'}{\theta}, \frac{(1-\epsilon)S'p'}{\theta}\right],\tag{22}$$

where S' = |S'|. To see this, first note that if S = 1 and $T_S < (1 - \epsilon)Sup'/\theta$, (22) automatically holds by letting S' = S. Suppose that $T_S < (1 - \epsilon)Sp'\theta$ for some S > 1 but $T_S < (1 - \epsilon)(S - 1)p'/\theta$. We then remove one peer from S and obtain S'. Clearly S' = |S'| = S - 1. We will have

$$T_{\mathcal{S}'} \le T_{\mathcal{S}} < (1-\epsilon)(S-1)p'/\theta = (1-\epsilon)S'p'\theta.$$

Hence, S' still satisfies $T_{S'} < \frac{(1-\epsilon)S'p'}{\theta}$. If (22) is still not true for S', we can remove another node from S' and repeat these steps until we find a set that satisfies (22). Note that by removing nodes one by one from S, in the worst case we will end up with a set S' that contains one peer. However, as mentioned above, if S' = |S'| = 1, (22) is automatically satisfied. As a result, we can always find a set S' that satisfies (22) by removing the nodes from S one by one. Therefore, the claim holds. Consequently,

$$\mathbf{P}\left(T_{\mathcal{S}} < \frac{(1-\epsilon)Sp'}{\theta} \text{ for some } \mathcal{S}\right)$$

$$\leq \mathbf{P}\left(T_{\mathcal{S}} \in \mathcal{I}_{\mathcal{S}}(\epsilon, p') \text{ for some } \mathcal{S}\right).$$
(23)

Now we are going to characterize the probability on the right hand side of (23). Define $r_i(S)$ to be the probability that a given user *i* select at least one of the peers in S as its downstream neighbor. For any peer $i \in S$, $r_i(S)$ is equal to 1 minus the probability that peer *i* choose all its *M* downstream neighbors from the peers that are not in S. More specifically, for $i \in S$ we have

$$r_i(\mathcal{S}) = \mathbf{P}\left(i \in \mathcal{U}(\mathcal{S})\right) = 1 - \frac{\binom{N-S}{M}}{\binom{N-1}{M}}.$$
(24)

Similarly, for $i \in V \setminus S$, we have

$$r_i(\mathcal{S}) = \mathbf{P}\left(i \in \mathcal{U}(\mathcal{S})\right) = 1 - \frac{\binom{N-S-1}{M}}{\binom{N-1}{M}}.$$

Note that for any peer *i*, the value of $r_i(S)$ is identical for all the sets S that have the same size |S|. In the rest of the proof, we will use $r_i(S)$ to denote the probability that user *i* selects at least one of the peers in S as its downstream neighbor for all the sets S that satisfies |S| = S, i.e., $r_i(S) = r_i(S), \forall S \subset V$ such that |S| = S. Note that,

$$1 - \frac{\binom{N-S}{M}}{\binom{N-1}{M}} \le 1 - \frac{\binom{N-S-1}{M}}{\binom{N-1}{M}}.$$

Thus, for any $i \in V$, (24) become a lower bound of $r_i(S)$.

$$r_i(S) \ge 1 - \frac{\binom{N-S}{M}}{\binom{N-1}{M}}.$$
 (25)

The second term on the right hand side of (25) satisfies

$$\frac{\binom{N-S}{M}}{\binom{N-1}{M}} = \frac{\frac{(N-S)!}{M!(N-S-M)}}{\frac{N!}{M!(N-M)!}} = \left(1 - \frac{M}{N}\right) \dots \left(1 - \frac{M}{N-S+1}\right)$$
$$\leq \left(1 - \frac{M}{N}\right)^S \leq e^{-\frac{SM}{N}}.$$

Combining (25) and the above inequality, we get a uniform lower bound of $r_i(S)$ for all *i*, which is denoted by r(S),

$$r_i(S) \ge 1 - e^{-\frac{SM}{N}} \triangleq r(S).$$

Now we have, for $y \ge (1-\epsilon)Np$

$$\begin{split} &\mathbf{P}\left(T_{\mathcal{S}} \in \mathcal{I}_{\mathcal{S}}(\epsilon, p') | Y = y\right) \\ \leq & \sum_{t = \lceil (1-\epsilon)(S-1)p'/\theta \rceil}^{\lfloor (1-\epsilon)Sp'/\theta \rfloor} \binom{y}{t} r(S)^t (1-r(S))^{y-t} \\ \leq & \frac{1}{\theta} \binom{y}{\lfloor (1-\epsilon)Sp'/\theta \rfloor} r(S)^{\lfloor (1-\epsilon)\frac{Sp'}{\theta} \rfloor} (1-r(S))^{y-\lfloor (1-\epsilon)\frac{Sp'}{\theta} \rfloor} \\ \leq & \frac{1}{\theta} N^{\frac{Sp'}{\theta}} e^{-\frac{S}{N}M(1-\epsilon)\left(Np-\frac{Sp'}{\theta}\right)}. \end{split}$$

Then, for $y \ge (1-\epsilon)Np$, we have

$$\begin{split} & \mathbf{P} \bigg(T_{\mathcal{S}} \in \mathcal{I}_{\mathcal{S}}(\epsilon, p') \text{ for some } S \leq \delta N \bigg| Y = y \bigg) \\ \leq & \sum_{S=1}^{\delta N} \binom{N}{S} \mathbf{P} \left(T_{\mathcal{S}} \in \mathcal{I}_{\mathcal{S}}(\epsilon, p') \middle| Y = y \right) \\ \leq & \sum_{S=1}^{\delta N} \frac{1}{\theta} N^{S} N^{\frac{Sp'}{\theta}} e^{-\frac{S}{N}M(1-\epsilon) \left(Np - \frac{Sp'}{\theta}\right)} \\ \leq & \sum_{S=1}^{\delta N} \frac{1}{\theta} N^{S} \binom{1+\frac{p'}{\theta}}{N} N^{-\alpha S(1-\epsilon) \left(p - \frac{p'\delta}{\theta}\right)} \text{ (since } M = \alpha \log N) \end{split}$$

It follows that

$$\mathbf{P}\left(T_{\mathcal{S}} \in \mathcal{I}_{\mathcal{S}}(\epsilon, p') \text{ for some } S \leq \delta N\right)$$

$$\leq \mathbf{P}(Y < (1-\epsilon)Np) + \sum_{\left[y=(1-\epsilon)Np\right]}^{N} \mathbf{P}(Y=y)$$

$$\times \mathbf{P}\left(T_{\mathcal{S}} \in \mathcal{I}_{\mathcal{S}}(\epsilon, p') \text{ for some } S \leq \delta N \middle| Y=y\right)$$

$$\leq O(\exp(-\epsilon^{2}p^{2}N)) + \sum_{S=1}^{\delta N} \frac{1}{\theta} N^{S(1+p'/\theta)} N^{-\alpha S(1-\epsilon)(p-p'\delta/\theta)}.$$
(26)

Note that when N is large, α satisfies,

$$\alpha > \frac{2 + (p + \epsilon)/\theta + d}{[p - (p + \epsilon)\delta/\theta](1 - \epsilon)} \geq \frac{2 + p'/\theta + d}{[p - p'\delta/\theta](1 - \epsilon)}$$

We have,

$$\sum_{S=1}^{\delta N} \frac{1}{\theta} N^{s(1+p'/\theta)} N^{-\alpha S(1-\epsilon)(p-p'\delta/\theta)p}$$

$$\leq \frac{1}{\theta} N^{1+1+p'/\theta} N^{-(2+p'/\theta+d)} = \frac{1}{\theta N^d} = O\left(\frac{1}{N^d}\right).$$
(27)

Finally, combining (21), (23), (26) and (27), we have

$$\begin{split} & \mathbf{P}\left(\bigcup_{\mathcal{S}\subset V, |\mathcal{S}|\leq\delta N}\Gamma_{\mathcal{S}}\right) \\ \leq & \mathbf{P}\left(T_{\mathcal{S}} < \frac{(1-\epsilon)sp'}{\theta} \text{ for some } S \leq \delta N\right) \\ \leq & \mathbf{P}\left(T_{\mathcal{S}} \in \mathcal{I}_{\mathcal{S}}(\epsilon,p') \text{ for some } S \leq \delta N\right) \\ \leq & O\left(\frac{1}{N^d}\right). \end{split}$$

2) When s is large, i.e., $S > \delta N$, the capacity from sophisticate allocation alone may not be adequate. We then need to count both parts of the capacity in (20). Consider the quantity $\theta uT_S + \sum_{i \in V} \sum_{j \in S} C_{ij}^U$ in (20). It can

be viewed as the maximum capacity that can be assigned to S from both the more sophisticate and uniform rate allocation. Now consider a purely uniform rate allocation. The total capacity allocated to S must be a lower bound of the above value. Next, we will show that the above lower bound will be larger than $(1 - \epsilon)SC_f$ with high probability. More precisely, let I_{ij} be the indicator function of the event that there is a link between node i and node j, and node i is an ON peer or the server. Then we have

$$\sum_{i \in V} \sum_{j \in \mathcal{S}} C_{ij}^U = \frac{u_s}{M} \sum_{j \in \mathcal{S}} I_{sj} + \frac{(1-\theta)u}{M} \sum_{i \in V \setminus s} \sum_{j \in \mathcal{S}} I_{ij}.$$

Note that for fixed $i \in V$, $\sum_{j \in S} I_{ij} \leq M$. Further, if i is OFF or $i \notin U(S)$, then $\sum_{j \in S} I_{ij} = 0$. Recall that T_S is the number of ON users in U(S). We have

$$\sum_{i \in V \setminus s} \sum_{j \in \mathcal{S}} I_{ij} = \sum_{i \in \mathcal{U}(\mathcal{S}), i \text{ is ON } j \in \mathcal{S}} \sum_{I_{ij} \leq T_{\mathcal{S}} M,$$

and hence

$$T_{\mathcal{S}} \ge \frac{1}{M} \sum_{i \in V \setminus s} \sum_{j \in \mathcal{S}} I_{ij}.$$

Then, the total available capacity from $\mathcal{U}(\mathcal{S})$ to \mathcal{S} will be

$$\theta u T_{\mathcal{S}} + \sum_{i \in V} \sum_{j \in \mathcal{S}} C_{ij}^{U}$$

$$\geq \frac{u_s}{M} \sum_{j \in \mathcal{S}} I_{sj} + \theta u \frac{1}{M} \sum_{i \in V \setminus s} \sum_{j \in \mathcal{S}} I_{ij} + \frac{(1 - \theta)u}{M} \sum_{i \in V} \sum_{j \in \mathcal{S}} I_{ij}$$

$$= \frac{u_s}{M} \sum_{j \in \mathcal{S}} I_{sj} + \frac{u}{M} \sum_{i \in V \setminus s} \sum_{j \in \mathcal{S}} I_{ij}.$$

The above value is equal to the capacity from $\mathcal{U}(S)$ to S if we use purely uniform rate allocation scheme. Note that

$$\mathbf{E}\left[\frac{u_s}{M}\sum_{j\in\mathcal{S}}I_{sj} + \frac{u}{M}\sum_{i\in V\setminus s}\sum_{j\in\mathcal{S}}I_{ij}\right]$$
$$= N\frac{M}{N}\cdot\frac{u_s}{M} + NS\frac{M}{N}\cdot\frac{u}{M}p \ge C_f.$$

Applying Chernoff bound and Lemma 5, and using similar argument as we did when proving Lemma 6, we can show that

$$\mathbf{P}(\Gamma_{\mathcal{S}}) = \mathbf{P}\left(\theta u T_{\mathcal{S}} + \sum_{i \in V} \sum_{j \in \mathcal{S}} C_{ij}^{U} \le (1 - \epsilon) SC_{f}\right)$$
$$\leq \mathbf{P}\left(\frac{u_{s}}{M} \sum_{j \in \mathcal{S}} I_{sj} + \frac{u}{M} \sum_{i \in V \setminus s} \sum_{j \in \mathcal{S}} I_{ij} \le (1 - \epsilon) SC_{f}\right)$$
$$< e^{-\frac{\epsilon^{2}}{2} \frac{u}{u_{s}} MSp}$$

Consequently,

$$\begin{split} &\mathbf{P}\left(\bigcup_{S\subset V, |S|>\delta N}\Gamma_{S}\right)\\ &\leq \sum_{S=\delta N+1}^{N}\binom{N}{s}e^{-\frac{\epsilon^{2}}{2}\frac{u}{u_{s}}MSp} \leq \sum_{S=\delta N+1}^{N}\left(\frac{Ne}{S}\right)^{S}e^{-\frac{\epsilon^{2}}{2}\frac{u}{u_{s}}MSp}\\ &\leq \sum_{S=\delta N+1}^{N}e^{S(1-\log\delta)}e^{-\frac{\epsilon^{2}}{2}\frac{u}{u_{s}}MSp} \leq 2e^{(1-\log\delta-\frac{\epsilon^{2}}{2}\frac{u}{u_{s}}Mp)\delta N}. \end{split}$$

Hence, as long as $1 - \log \delta - \frac{\epsilon^2}{2} \frac{u}{u_s} Mp < 0$, the above expression will converge to 0 exponentially fast. In fact, if $M = \alpha \log N$ and α satisfies (14), then for sufficiently large N, the inequality $1 - \log \delta - \frac{\epsilon^2}{2} Mp < 0$ always holds. Hence, if (14) holds, we have,

$$\mathbf{P}\left(\bigcup_{\mathcal{S}\subset V, |\mathcal{S}| > \delta N} \Gamma_{\mathcal{S}}\right) \le O(1/N^d)$$

Finally, by combining the result of part (1) and part (2) together, we can thus prove the lemma.

APPENDIX F

PROOF OF THEOREM 8

Proof: Recall that for any fixed destination t, H_t denotes the sub-network of the system that only contains the ON peers, and D_m is the capacity of the cut W_m in H_t that have m ON peers on and the server on the left and the destination t and all other ON peers on the right. Let \mathcal{B}'_m denote the event $\{D_m \leq (1-\epsilon)C_f \text{ for any cut} among the <math>\binom{Y}{m}$ cuts $\}$. Using Proposition 2, we have

$$\mathbf{P}\left(C_{\min-\min}(s \to V) \leq (1-\epsilon)C_{f}\right)$$

$$\leq \sum_{y=0}^{N-1} \mathbf{P}\left(C_{\min-\min}(s \to V) \leq (1-\epsilon)C_{f} | Y = y\right) \mathbf{P}(Y = y)$$

$$= \sum_{y=0}^{N-1} \mathbf{P}\left(\bigcup_{t \in V} \bigcup_{m=0}^{Y} \mathcal{B}'_{m} \middle| Y = y\right) \mathbf{P}(Y = y)$$

$$\leq \sum_{y=0}^{N-1} \mathbf{P}\left(\bigcup_{t \in V} \left\{D_{Y} \leq (1-\epsilon)C_{f}\right\} \middle| Y = y\right) \mathbf{P}(Y = y)$$

$$+ \sum_{y=0}^{N-1} \mathbf{P}\left(\bigcup_{t \in V} \bigcup_{m=0}^{Y-1} \mathcal{B}'_{m} \middle| Y = y\right) \mathbf{P}(Y = y).$$
(28)

The first term in the last step of (28) corresponds to the "last hop". It is the probability that the capacity that some peer receives from its direct upstream neighbors is smaller than $(1 - \epsilon)C_f$, or equivalently, the probability that we cannot find C_{ij}^S 's such that (10) holds. According to Corollary 10, for the hybrid scheme π_{θ} , this event is equivalent to the event that for some subset S of V, the total allocated capacity from their upstream neighbors is smaller than $|S|(1-\epsilon)C_f$. Then, by Lemma 11, the probability of such an event is $O\left(\frac{1}{N^d}\right)$. More specifically, if (14) holds, we have

$$\sum_{y=0}^{N-1} \mathbf{P}\left(\bigcup_{t\in V} \left\{ D_Y \le (1-\epsilon)C_f \right\} \middle| Y = y \right) \mathbf{P}(Y=y)$$
$$= \mathbf{P}\left(\bigcup_{t\in V} \left\{ \sum_{i\in V} C_{it} \le (1-\epsilon)C_f \right\} \right)$$

=**P** (There do not exist C_{ij}^S 's such that (10) holds)

$$= \mathbf{P}\left(\sum_{i \in \mathcal{U}(\mathcal{S})} \theta u_i \ge \sum_{j \in \mathcal{S}} \left((1 - \epsilon)C_f - \sum_{i \in \mathcal{U}(j)} C_{ij}^U \right) \right)$$
$$\le O\left(\frac{1}{N^d}\right),$$

where the second last step comes from Corollary 10 and the last step comes from Lemma 11.

The second term in the last step of (28) represents the probability that some cut other than the "last hop" does not have sufficient capacity. The capacity from the more sophisticated allocation is difficult to characterize for non-last-hop cuts. Fortunately, since we have at least two peers on the right side of the cut, the capacity from the uniform allocation alone will be sufficient. Define $\eta = \max\{1 - \epsilon, \frac{1}{2\theta}\}$. It is not hard to see that if $\theta > 1/2$, then for all $Y \ge \eta p(N-1) + 1$, $m \le Y - 1$ and large enough N,

$$\begin{split} \mathbf{E}[D_m] \geq & \frac{u_s(Y-m)}{Y} + \theta \frac{u}{N-1} m(Y-m+1) \\ \geq & \min\{2\theta\eta u_s, \frac{u_s}{N} + 2\theta\eta up\} \\ = & \max\{2(1-\epsilon)\theta, 1\}C_f. \end{split}$$

Hence, the expectation of the cut capacity D_m is always larger than the optimal streaming rate C_f . Therefore,

$$\mathbf{P} \left(D_m \le (1-\epsilon)C_f | Y = y \right)$$

$$\le \mathbf{P} \left(D_m \le \frac{(1-\epsilon)}{\max\{2(1-\epsilon)\theta, 1\}} \mathbf{E}[D_m] \middle| Y = y \right)$$

$$\le \mathbf{P} \left(D_m \le \min\{1-\epsilon, \frac{1}{2\theta}\} \mathbf{E}[D_m] \middle| Y = y \right).$$

Now apply Lemma 4 and replace ϵ by $\min\{1-\epsilon, \frac{1}{2\theta}\}$. We then have

$$\begin{split} \mathbf{P}\left(D_m &\leq \min\{1-\epsilon, \frac{1}{2\theta}\}\mathbf{E}[D_m] \middle| Y = y\right) \\ \leq & \exp\left[-\frac{u}{u_s} \max\left\{\frac{\epsilon^2}{2}, \frac{(2\theta-1)^2}{8\theta^2}\right\} \\ & \times \left(Mm\frac{y-m+1}{N-1} + M\frac{y-m}{y}\right)\right]. \end{split}$$

November 19, 2012

Let

$$\beta' = \exp\left(-\frac{u}{u_s} \max\left\{\frac{\epsilon^2}{2}, \frac{(2\theta - 1)^2}{8\theta^2}\right\}\right).$$

Applying Lemme 7, we have

$$\sum_{y=0}^{N-1} \mathbf{P}\left(\bigcup_{m=0}^{Y-1} \mathcal{B}'_m \middle| Y = y\right) \mathbf{P}(Y = y) = \mathbf{P}\left(\bigcup_{m=0}^{Y-1} \mathcal{B}'_m\right)$$
$$\leq O(\exp(-(1-\eta)^2 p^2 N)) + \beta'^{\eta p} \left[\left(1+\beta'^{\frac{\eta p}{2}}\right)^{N-1} - 1\right].$$

Then follow the same approach as in the proof of Theorem 1, one can show that if $M = \alpha \log N$ and

$$\alpha \ge \frac{(2+d)u_s}{\eta pu \max\left\{\frac{\epsilon^2}{2}, \frac{(2\theta-1)^2}{8\theta^2}\right\}},\tag{29}$$

then

$$\sum_{y=0}^{N-1} \mathbf{P}\left(\bigcup_{m=0}^{Y-1} \mathcal{B}'_m \middle| Y = y\right) \mathbf{P}(Y = y) \le O\left(\frac{1}{N^{d+1}}\right).$$

It follows that the second term in (28) can be bounded by

$$\sum_{y=0}^{N-1} \mathbf{P}\left(\bigcup_{t \in V} \bigcup_{m=0}^{Y-1} \mathcal{B}'_m \middle| Y = y\right) \mathbf{P}(Y = y)$$
$$\leq NO\left(\frac{1}{N^{d+1}}\right) = O\left(\frac{1}{N^d}\right).$$

In conclusion, if (14) and (29) hold, i.e., (11) holds then

$$\mathbf{P}\left(C_{\min-\min}(s \to V) \le (1-\epsilon)C_f\right) \le O\left(\frac{1}{N^d}\right)$$

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APPENDIX G

PROOF OF LEMMA 15

Proof: We first claim that for any $\theta > 0$, the moment generating function of any bounded random variable U between $[0, U_{\text{max}}]$ with the mean value μ will be smaller than the moment generating function of the "ON-OFF" random variable \tilde{U} with the same expectation μ , where $\mathbf{P}(\tilde{U} = U_{\text{max}}) = \frac{\mu}{U_{\text{max}}}$ and $\mathbf{P}(\tilde{U} = 0) = 1 - \frac{\mu}{U_{\text{max}}}$. More specific, we claim the following inequality holds for any $\theta > 0$

$$\mathbf{E}[e^{-\theta U}] \le \mathbf{E}[e^{-\theta \tilde{U}}] = \frac{\mu}{U_{\max}} e^{-\theta U_{\max}} + 1 - \frac{\mu}{U_{\max}}.$$

To see this, note that for $\theta = 0$,

$$\mathbf{E}[e^{-\theta U}] = \mathbf{E}[e^{-\theta U}] = 1.$$

For any $\theta > 0$

$$\frac{d}{d\theta}\mathbf{E}[e^{-\theta U}] = \mathbf{E}[-Ue^{-\theta U}] \le \mathbf{E}[-Ue^{-\theta U_{\max}}] = -\mu e^{-\theta U_{\max}}.$$

On the other hand,

$$\begin{aligned} \frac{d}{d\theta} \mathbf{E}[e^{-\theta \tilde{U}}] &= \frac{d}{d\theta} \mathbf{E} \left[\frac{\mu}{U_{\max}} e^{-\theta U_{\max}} + 1 - \frac{\mu}{U_{\max}} \right] \\ &= -\mu e^{-\theta U_{\max}}. \end{aligned}$$

Thus, for any $\theta > 0$

$$\frac{d}{d\theta}(\mathbf{E}[e^{-\theta U}] - \mathbf{E}[e^{-\theta \tilde{U}}]) \le 0.$$

Consequently, for any $\theta > 0$

$$\mathbf{E}[e^{-\theta U}] \le \mathbf{E}[e^{-\theta \tilde{U}}].$$

Now, by Chernoff bounds, we have for, $\theta > 0$,

$$\mathbf{P}(C_{\tilde{k}} \le (1-\epsilon)\mathbf{E}[C_{\tilde{k}}]) \le \frac{\mathbf{E}[e^{-\theta C_{\tilde{k}}}]}{e^{-\theta \mathbf{E}[C_{\tilde{k}}]}}.$$
(30)

We are going to find an upper bound on the moment generating function of $C_{\tilde{k}}$, $\mathbf{E}[e^{-\theta C_{\tilde{k}}}]$, as we did in Proposition 5.

$$\mathbf{E}[e^{-\theta C_{\tilde{k}}}] = \mathbf{E}\left[e^{-\theta \sum_{i \in V_{\tilde{k}}} \sum_{j \in V_{\tilde{k}}^c} C_{ij}}\right]$$
$$= \left[\mathbf{E}\left[e^{-\theta \sum_{j \in V_{\tilde{k}}^c} C_{1j}}\right]\right]^{\tilde{k}}.$$
(31)

Recall that I_{ij} is the indicator function of the event that peer j is a downstream neighbors of i, and \tilde{I}_{ij} , $j \in V_{\tilde{k}}^c$ are i.i.d. Bernoulli random variables such that \tilde{I}_{ij} has the same marginal distribution as I_{ij} . Then, according to

$$\mathbf{E}\left[e^{-\theta\sum_{j\in V_{k}^{c}}C_{1j}}\right]$$

$$=\mathbf{E}\left[\left(e^{-\theta\sum_{j\in V_{k}^{c}}I_{1j}}\right)^{\frac{U_{1}}{M}}\right]$$

$$=\mathbf{E}\left[\mathbf{E}\left[\left(e^{-\theta\sum_{j\in V_{k}^{c}}I_{1j}}\right)^{\frac{U_{1}}{M}}\middle|U_{1}\right]\right]$$

$$\leq\mathbf{E}\left[\mathbf{E}\left[\left(e^{-\theta\sum_{j\in V_{k}^{c}}\tilde{I}_{1j}}\right)^{\frac{U_{1}}{M}}\middle|U_{1}\right]\right] \quad (by \text{ Theorem 4})$$

$$=\mathbf{E}\left[\mathbf{E}\left[\left(1-\frac{M}{N-1}+\frac{M}{N-1}e^{-\theta\frac{U_{1}}{M}}\right)^{N-k}\middle|U_{1}\right]\right]$$

$$=\mathbf{E}\left[\left(1-\frac{M}{N-1}+\frac{M}{N-1}e^{-\theta\frac{U_{1}}{M}}\right)^{N-\tilde{k}}\right]. \quad (32)$$

Note that according to our claim, letting $\tilde{p} = \frac{\mu}{U_{\text{max}}}$ and $q = \frac{M}{N-1}$, we have,

$$\mathbf{E}\left[\left(1-\frac{M}{N-1}+\frac{M}{N-1}e^{-\theta\frac{U_{1}}{M}}\right)^{N-\tilde{k}}\right] \\
=\sum_{l=0}^{N-\tilde{k}}\binom{N-\tilde{k}}{l}(1-q)^{N-\tilde{k}-l}q^{l}\mathbf{E}\left[e^{-\theta\frac{U_{1}}{M}l}\right] \\
\leq\sum_{l=0}^{N-\tilde{k}}\binom{N-\tilde{k}}{l}(1-q)^{N-\tilde{k}-l}q^{l}\mathbf{E}\left[e^{-\theta\frac{\tilde{U}}{M}l}\right] \\
=\sum_{l=0}^{N-\tilde{k}}\binom{N-\tilde{k}}{l}(1-q)^{N-\tilde{k}-l}q^{l}\left(pe^{-\theta\frac{U_{\max}}{M}l}+1-\tilde{p}\right) \\
=\tilde{p}\left(1-q+qe^{-\theta\frac{U_{\max}}{M}}\right)^{N-\tilde{k}}+1-\tilde{p}.$$
(33)

Now, combine (31), (32) and (33) we get

$$\mathbf{E}[e^{-\theta C_{\tilde{k}}}] \leq \left[\tilde{p}\left(1 - q + qe^{-\theta \frac{U_{\max}}{M}}\right)^{N-\tilde{k}} + 1 - \tilde{p}\right]^{\tilde{k}}.$$
(34)

Hence, (30) can be written as

$$\begin{split} \mathbf{P}(C_{\tilde{k}} &\leq (1-\epsilon)\mathbf{E}[C_{\tilde{k}}]) \leq \frac{\mathbf{E}[e^{-\theta C_{\tilde{k}}}]}{e^{-\theta \mathbf{E}[C_{\tilde{k}}]}} \\ &\leq \left[\tilde{p}\left(1-q+qe^{-\theta \frac{U_{\max}}{M}}\right)^{N-\tilde{k}}+1-\tilde{p}\right]^{\tilde{k}} e^{\theta(1-\epsilon)\mu \frac{\tilde{k}(N-\tilde{k})}{N-1}}. \end{split}$$

Pick $\theta = -\frac{M}{U_{\text{max}}}\log(1-\epsilon)$, we get

$$\mathbf{P}(C_{\tilde{k}} \leq (1-\epsilon)\mathbf{E}[C_{\tilde{k}}]) \\
\leq \frac{\mathbf{E}[e^{-\theta C_{\tilde{k}}}]}{e^{-\theta \mathbf{E}[C_{\tilde{k}}]}} \\
\leq \left[\tilde{p}\left(1-q+q(1-\epsilon)\right)^{N-\tilde{k}}+1-\tilde{p}\right]^{\tilde{k}}e^{-(1-\epsilon)\log(1-\epsilon)\tilde{p}M\frac{\tilde{k}(N-\tilde{k})}{N-1}} \\
= \left[\tilde{p}\left(1-\epsilon q\right)^{N-\tilde{k}}+1-\tilde{p}\right]^{\tilde{k}}e^{-(1-\epsilon)\log(1-\epsilon)\tilde{p}q\tilde{k}(N-\tilde{k})} \\
\leq \exp\left(\tilde{p}[(1-\epsilon q)^{(N-\tilde{k})}-1]\tilde{k}-(1-\epsilon)\log(1-\epsilon)\tilde{p}q\tilde{k}(N-\tilde{k})\right).$$
(35)

Note that $N - \tilde{k} < 2/p$ and it will not scale with N. The Taylor expansion of $(1 - \epsilon q)^{(N - \tilde{k})}$ around q = 0 is

$$(1 - \epsilon q)^{(N-k)} = 1 - \epsilon (N - \tilde{k})q + O(q^2).$$

The exponent of (35) would be

$$\begin{split} \tilde{p}\tilde{k} \left[(1-\epsilon q)^{(N-\tilde{k})} - 1 - (N-\tilde{k})q(1-\epsilon)\log(1-\epsilon) \right] \\ = \tilde{p}\tilde{k} \left[1 - \epsilon(N-\tilde{k})q + O(q^2) - 1 - (N-\tilde{k})q(1-\epsilon)\log(1-\epsilon) \right] \\ \leq \tilde{p}\tilde{k} \left[\epsilon(N-\tilde{k})q + O(q^2) - (N-\tilde{k})q\left(\frac{\epsilon^2}{2} - \epsilon\right) \right] \\ = - \tilde{p}\tilde{k} \left[(N-\tilde{k})q\frac{\epsilon^2}{2} + O(q^2) \right]. \end{split}$$

Finally, the following holds,

$$\mathbf{P}(C_{\tilde{k}} \le (1-\epsilon)\mathbf{E}[C_{\tilde{k}}]) \le e^{-\tilde{p}\tilde{k}\left[(N-\tilde{k})q\frac{\epsilon^2}{2} + O(q^2)\right]}.$$

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APPENDIX H

PROOF OF THEOREM 12

Proof: As discussed before, we will divide all the cuts into three different groups: 1)where the capacities between ON peers is sufficient, 2)where the capacities from the ON peers to all kinds of peers is adequate and

3) where the capacities from the ONE peers alone is insufficient. More specifically, let $\epsilon' = 1 - \sqrt{1 - \epsilon}$, we have

$$\mathbf{P}(C_{\min-\min}(s \to V) \le (1-\epsilon)C_f) \le O(\exp(-\epsilon'^2 p^2 (N-1))) + \sum_{y=(1-\epsilon')p(N-1)}^N \mathbf{P}(Y=y)$$

$$\times \left[\sum_{m=0}^{s} \binom{y}{m} \mathbf{P}(D_m \le (1-\epsilon)C_f | Y = y)\right]$$

$$N-m-\frac{2}{2}-1$$
(36)

$$+\sum_{m=y-\frac{2}{p}}^{y} \binom{y}{m} \sum_{k=0}^{N-m-\frac{1}{p^{2}}-1} \binom{N-y}{k} \mathbf{P}(D_{m,k} \le (1-\epsilon)C_{f}|Y=y)$$
(37)

$$+\sum_{m=y-\frac{2}{p}}^{y} {\binom{y}{m}} \sum_{k=N-m-\frac{2}{p^2}}^{N-y} {\binom{N-y}{k}} \mathbf{P}(C_{m,k} \le (1-\epsilon)C_f | Y=y) \bigg].$$
(38)

For (36), by Lemma 13, we have for $y \ge (1 - \epsilon')p(N - 1)$

$$\mathbf{P} \left(D_m \le (1 - \epsilon) C_f | Y = y \right)$$
$$\le \mathbf{P} \left(\tilde{D}_m \le (1 - \epsilon') \mathbf{E} [\tilde{D}_m] | Y = y \right).$$

Thus, when $m \leq y - \frac{2}{p}$, we have

$$\sum_{m=0}^{y-\frac{2}{p}} {\binom{y}{m}} \mathbf{P}(D_m \le (1-\epsilon)C_f | Y = y)$$
$$\le \sum_{m=0}^{y-\frac{2}{p}} {\binom{y}{m}} \mathbf{P}(\tilde{D}_m \le (1-\epsilon')\mathbf{E}[\tilde{D}_m] | Y = y)$$

Recall that D_m is capacity of a cut in the sub-network that contains only the ON peers of the network where the upload capacity of each peer is given by a ON-OFF model. By our previous result, when $\alpha \ge \frac{4du_s}{(1-\epsilon')\epsilon'^2 pu}$, we have

$$\sum_{y=(1-\epsilon')pN}^{N}\sum_{m=0}^{y-\frac{2}{p}} \binom{y}{m} \mathbf{P}(\tilde{D}_m \le (1-\epsilon')\mathbf{E}[\tilde{D}_m]|Y=y)\mathbf{P}(Y=y) < O\left(\frac{1}{N^{2d}}\right).$$

For (37), by Lemma 14, we have

$$\mathbf{P} \left(D_{m,k} \le (1-\epsilon)C_f | Y = y \right)$$
$$\le \mathbf{P} \left(\tilde{D}_{m,k} \le (1-\epsilon')\mathbf{E}[\tilde{D}_{m,k}] | Y = y \right).$$

Thus, when $m \ge Y - \frac{2}{p}$ and $N - k - m > \frac{2}{p^2}$, we have

$$\sum_{m=y-\frac{2}{p}}^{y} {\binom{y}{m}} \sum_{k=0}^{N-m-\frac{2}{p^{2}}-1} {\binom{N-y-1}{k}} \mathbf{P}(D_{m,k} \le (1-\epsilon)C_{f}|Y=y)$$
$$\le \sum_{m=y-\frac{2}{p}}^{y} {\binom{y}{m}} \sum_{k=0}^{N-m-\frac{2}{p^{2}}-1} {\binom{N-y-1}{k}} \mathbf{P}(\tilde{D}_{m,k} \le (1-\epsilon')\mathbf{E}[\tilde{D}_{m,k}]|Y=y)$$

Using Proposition 3 and Lemma 4, one can show that

$$\mathbf{P}(\tilde{D}_{m,k} \le (1-\epsilon')\mathbf{E}[\tilde{D}_{m,k}]|Y=y) \le e^{-\frac{\epsilon^2}{2}\frac{m(N-k-m)}{N-1}M}$$

Note that for $m \ge y - \frac{2}{p}$, when N is sufficiently large and so is y, we have $m \ge (1 - \epsilon')pN$. Then (37) become

$$\begin{split} &\sum_{y=(1-\epsilon')pN}^{N}\sum_{m=y-\frac{2}{p}}^{y}\binom{y}{m}\sum_{k=0}^{N-m-\frac{2}{p^{2}}-1}\binom{N-y-1}{k}\mathbf{P}(D_{m,k}\leq(1-\epsilon)C_{f}|Y=y)\mathbf{P}(Y=y) \\ &\leq \sum_{y=(1-\epsilon')pN}^{N}\sum_{m=y-\frac{2}{p}}^{y}\binom{y}{m}\sum_{k=0}^{N-m-\frac{2}{p^{2}}-1}\binom{N-y-1}{k}e^{-\frac{\epsilon'^{2}}{2}\frac{(y-\frac{2}{p})(N-k-m)}{N-1}}M\mathbf{P}(Y=y) \\ &\leq \sum_{y=(1-\epsilon')pN}^{N}\sum_{m=0}^{y}\binom{y}{m}\sum_{k=0}^{N-y-1}\binom{N-y-1}{k}e^{-\frac{\epsilon'^{2}}{2}\frac{(1-\epsilon')pN(N-k-m)}{N-1}}M\mathbf{P}(Y=y) \\ &\leq \sum_{y=(1-\epsilon')pN}^{N}\mathbf{P}(Y=y)e^{-(1-\epsilon')\frac{\epsilon'^{2}}{2}pM}\sum_{m=0}^{y}\binom{y}{m}e^{-(1-\epsilon')\frac{\epsilon'^{2}}{2}p(Y-m)M} \\ &\times \sum_{k=0}^{N-y-1}\binom{N-y-1}{k}e^{-(1-\epsilon')\frac{\epsilon'^{2}}{2}p(N-y-1-k)M}e^{-(1-\epsilon')\frac{\epsilon'^{2}}{2}p(N-y-1-k)M}e^{-(1-\epsilon')\frac{\epsilon'^{2}}{2}pM}\binom{y}{1+e^{-(1-\epsilon')\frac{\epsilon'^{2}}{2}pM}}^{N-y-1} \\ &\leq e^{-(1-\epsilon')pN}\mathbf{P}(Y=y)e^{-(1-\epsilon')\frac{\epsilon'^{2}}{2}pM}\binom{1+e^{-(1-\epsilon')\frac{\epsilon'^{2}}{2}pM}}{1+e^{-(1-\epsilon')\frac{\epsilon'^{2}}{2}pM}}^{N-y-1}. \end{split}$$

Clearly, if $\alpha \geq \frac{4du_s}{(1-\epsilon')\epsilon'^2 pu}$, the above expression will be

$$e^{-(1-\epsilon')\frac{\epsilon'^2}{2}pM} \left(1+e^{-(1-\epsilon')\frac{\epsilon'^2}{2}pM}\right)^{N-1} \le N^{-2d}(1+N^{-2d})^{N-1}$$
$$\le O\left(\frac{1}{N^{2d}}\right).$$

For (38), we have

$$\begin{split} &\sum_{y=(1-\epsilon')pN}^{N} \mathbf{P}(Y=y) \sum_{m=y-\frac{2}{p}}^{y} \binom{y}{m} \sum_{k=N-m-\frac{2}{p^{2}}}^{N-y-1} \binom{N-y}{k} \mathbf{P}(C_{m,k} \leq (1-\epsilon)C_{f}|Y=y) \\ &= \sum_{m,k:N-m-k \leq 2/p^{2}} \binom{y}{m} \binom{N-y-1}{k} \mathbf{P}(C_{m,k} \leq (1-\epsilon)C_{f}|Y=y) \\ &\leq \sum_{y=(1-\epsilon')pN}^{N} \mathbf{P}(Y=y) \sum_{m,k:N-m-k \leq 2/p^{2}} \binom{N}{m+k} \mathbf{P}(C_{m+k} \leq (1-\epsilon)C_{f}|Y=y) \\ &= \sum_{\tilde{k}=N-2/p^{2}}^{N-1} \binom{N-1}{\tilde{k}} \mathbf{P}(C_{\tilde{k}} \leq (1-\epsilon)C_{f}) \qquad (\tilde{k}=m+k). \end{split}$$

By Lemma 15, we have

$$\sum_{\tilde{k}=N-2/p^{2}}^{N-1} {\binom{N-1}{\tilde{k}}} \mathbf{P}(C_{\tilde{k}} \leq (1-\epsilon)C_{f})$$

$$\leq \sum_{\tilde{k}=N-2/p^{2}}^{N-1} {\binom{N-1}{\tilde{k}}} e^{-\tilde{p}\tilde{k}\left[(N-\tilde{k})q\frac{\epsilon^{2}}{2}+O(q^{2})\right]}$$

$$= \sum_{\tilde{k}=N-2/p^{2}}^{N-1} {\binom{N-1}{\tilde{k}}} e^{-\tilde{p}\tilde{k}\left[\frac{(N-\tilde{k})M}{N-1}\frac{\epsilon^{2}}{2}+O\left(\frac{M^{2}}{(N-1)^{2}}\right)\right]}$$

$$\leq e^{O\left(\frac{M^{2}}{N-1}\right)} e^{-\tilde{p}M\frac{\epsilon^{2}}{2}} \sum_{\tilde{k}=N-2/p^{2}}^{N-1} {\binom{N}{\tilde{k}}} e^{-\tilde{p}\frac{\epsilon^{2}}{2}(N-1-\tilde{k})M}$$

$$\leq e^{O\left(\frac{M^{2}}{N-1}\right)} e^{-\tilde{p}\frac{\epsilon^{2}}{2}M} \left(1+e^{-\tilde{p}\frac{\epsilon^{2}}{2}M}\right)^{N-1}$$

If $\alpha \geq \frac{4d}{\epsilon^2 \tilde{p}}$, then

$$e^{-\tilde{p}\frac{\epsilon^2}{2}M} \le \frac{1}{N^{2d}},$$

and

$$\begin{split} & e^{O\left(\frac{M^2}{N-1}\right)} e^{-\tilde{p}\frac{\epsilon^2}{2}M} \left(1+e^{-\tilde{p}\frac{\epsilon^2}{2}M}\right)^{N-1} \\ & \leq O(1)\frac{1}{N^{2d}} \left(1+\frac{1}{N^{2d}}\right)^{N-1} \\ & \leq O\left(\frac{1}{N^{2d}}\right). \end{split}$$

By combining the three parts of results together, we can conclude that if

$$\alpha \geq \max\left\{\frac{4d}{\epsilon^2 \tilde{p}}, \frac{4du_s}{(1-\epsilon')\epsilon'^2 pu}, \right\}$$

then

$$\mathbf{P}(C_{\min-\min}(s \to V) \le (1-\epsilon)C_f) \le O\left(\frac{1}{N^{2d}}\right).$$