# Performance Limits of Greedy Maximal Matching in Multi-hop Wireless Networks

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**Abstract**

In this paper, we characterize the performance limits of an important class of scheduling schemes, called *Greedy Maximal Matching* (*GMM*), for multi-hop wireless networks. For simplicity, we focus on the well-established *node-exclusive* interference model, although many of the stated results can be readily extended to more general interference models. The study of the performance of *GMM* is intriguing because although a lower bound on its performance is well known, empirical observations suggest that this bound is quite loose, and that the performance of *GMM* is often close to optimal. In fact, recent results have shown that *GMM* achieves optimal performance under certain conditions. In this paper, we provide new analytic results that characterize the performance of *GMM* through the topological properties of the underlying graphs. To that end, we generalize a recently developed topological notion called the *local pooling* condition to a far weaker condition called the $\sigma$*-local pooling*. We then define the *local-pooling factor* on a graph, as the supremum of all $\sigma$ such that the graph satisfies $\sigma$-local pooling. We show that for a given graph, the efficiency ratio of *GMM* (i.e., the worst-case ratio of the throughput of *GMM* to that of the optimal) is equal to its local-pooling factor. Further, we provide results on how to estimate the local-pooling factor for arbitrary graphs and show that the efficiency ratio of *GMM* is no smaller than $d^*/(2d^*-1)$ in a network topology of maximum node-degree $d^*$. We also identify specific network topologies for which the efficiency ratio of *GMM* is strictly less than 1.

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# I. INTRODUCTION

Over the last few years there has been significant interest in studying the *scheduling* problem for multi-hop wireless networks [1]–[11]. In general, this problem involves determining which links should transmit (i.e., which node-pairs should communicate) and at what times, what modulation and coding schemes should be used, and at what power levels should communication take place. While the optimal solution of this scheduling problem has now been known for a long time [2], the resultant solution is extremely complex and virtually impossible to implement.

Recently, several researchers have developed simpler scheduling solutions for an important class of interference models called the *node-exclusive* (or *primary* interference model. Under this interference model, a node cannot simultaneously transmit or receive, and cannot simultaneously communicate with two or more nodes in the network. The node-exclusive model is a good representation for practical wireless systems using Bluetooth or FH-CDMA networks [1], [12], [13]. Under this model, the scheduling problem can be mapped to a *matching* problem, i.e., any active set of links must form a *matching* of the nodes in the network. In this setting, there exists a polynomial-time optimal solution called the *Maximum Weighted Matching* (*MWM*) policy. However, the complexity of *MWM* is roughly $O(N^3)$ [14], where $N$ is the total number of nodes in the network. Hence, it is still too complex to implement in most practical scenarios.

To address this issue, a well-known suboptimal solution called the *Greedy Maximal Matching* (*GMM*) has been developed that significantly reduces the scheduling complexity [1], [15], [16]. We can characterize the performance of *GMM* through its efficiency ratio $\gamma^*$, which is the largest number $\gamma$ such that for any offered load $\vec{\lambda}$ that the optimal *MWM* policy can support, *GMM* can support $\gamma\vec{\lambda}$. It is relatively straightforward to show that the efficiency ratio of *GMM* is at least $1/2$, i.e., *GMM* can sustain at least half of the throughput of the optimal *MWM* policy. In fact, simulation results suggest that the performance of *GMM* is often much better than this lower bound in most network settings. Further, it has been shown recently that if the network topology satisfies the so-called *local pooling* condition [17], [18], then *GMM* can in fact achieve the full capacity region. Unfortunately, realistic network topologies may not satisfy the local pooling condition, and hence their true efficiency ratio remains unknown.

In this paper, our main contribution is to provide new analytical results on the achievable efficiency ratio of *GMM* for a large class of network topologies. Such an evaluation is important for the following reasons:

- It has been empirically observed in [3] that the throughput achieved by *GMM* is virtually the same as the maximum achievable throughput for a variety of networking scenarios.

- *GMM* can be implemented in a distributed manner [19], which is critical from the point of view of many multi-hop networking systems. Further, even simpler constant-time-complexity random algorithms can be developed to approximate the performance of *GMM* [3].
- Although many distributed scheduling schemes have been recently developed [7], [9]–[11], the study of *GMM* continues to remain attractive because, empirically, *GMM* performs better than these schemes [8], either in terms of the achievable throughput, or in terms of the resultant queueing delay.

In this paper, we provide two main results along this direction. First, we generalize the notion of local pooling in [17], and derive an equivalent characterization of the efficiency ratio of *GMM* through a topological property, i.e., the *local-pooling factor*, of the underlying network graph. In particular, we show that the efficiency ratio of *GMM* under a given network topology is equal to its local-pooling factor. Second, we provide preliminary results for estimating the local-pooling factor of arbitrary network graphs. Using these results, we are able to identify network topologies where the efficiency ratio of *GMM* could be low.

The rest of the paper is organized as follows. We first describe our model in Section II. We then introduce the notion of local-pooling factor and show in Section III that the efficiency ratio of *GMM* under an arbitrary network topology is equal to its local-pooling factor. In Section IV, by characterizing the set of unstable links under *GMM*, we provide preliminary results for estimating the local-pooling factors under arbitrary network graphs. These results lead to the discovery of network graphs where the efficiency ratio of *GMM* is low. We conclude in Section V.

## II. NETWORK MODEL

We model a wireless network by a graph $G(V, E)$, where $V$ is the set of nodes, and $E$ is the set of undirected links. We assume a time-slotted system, where the length of each time slot is of unit length. We assume that in each time slot, a link can transmit one packet provided that the following *node-exclusive interference* constraint is satisfied: if a link $l$ is transmitting data, then no other links that share a common transmitter node or receiver node with link $l$ can transmit at the same time. Hence, any active set of links must form a *matching* of the nodes in $V$.

Let $M_E$ be a *maximal* matching on $E$, i.e., no more links can be added to $M_E$ without violating the node-exclusive interference constraint. We use a vector in $\{0,1\}^{|E|}$ to denote a maximal matching $M_E$ such that the $k$-th element is set to 1 if link $k \in E$ is included in the maximal matching $M_E$, and to 0 otherwise. Let $\mathcal{M}_E$ be the set of all possible maximal matchings and let $Co(\mathcal{M}_E)$ denote its convex hull.

We assume that packets arrive to each link $l$ according to a stationary and ergodic process, and the average arrival rate is $\lambda_l$. The *capacity region* (or the *stability region*) under a given scheduling policy is defined as the set of arrival rate vectors $\vec{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_E]$ such that the system is stable (i.e., all queues are kept finite). It can be shown that the *optimal* capacity region $\Lambda$ is given by [2], [20]–[22],

$$\Lambda = \left\{ \vec{\lambda} \mid \vec{\lambda} \preceq \vec{\phi}, \text{ for some } \vec{\phi} \in Co(\mathcal{M}_E) \right\}, \tag{1}$$

where $\vec{x} \preceq \vec{y}$ denotes that $\vec{x}$ is component-wise dominated by $\vec{y}$. It is well-known that the *Maximal Weighted Matching (MWM)* policy can achieve this optimal capacity region under the node-exclusive interference model. However, its computational complexity ($O(N^3)$) is high. In this paper, we are interested in a suboptimal (but much simpler) policy called *Greedy Maximal Matching* (*GMM*). *GMM* operates as follows: at each time slot, it first picks the link $l$ with the largest backlog; it then discards all links that interfere with link $l$; it then picks the link with the largest backlog from the remaining links; and this process continues until no links are left. As we discussed in the introduction, in this paper we are interested in characterizing the efficiency ratio of *GMM* under arbitrary network topologies. We formally define the notion of efficiency ratio as follows.

*Definition 1:* For a suboptimal scheduling policy, e.g., *GMM*, we say that it achieves a *fraction* $\gamma$ *of the capacity region* under a given network topology if it can keep the system stable for any offered load $\vec{\lambda} \in \gamma\Lambda$.

*Definition 2:* The *efficiency ratio* $\gamma^*$ of a scheduling policy under a given network topology is the supremum of all $\gamma$ such that the policy can achieve a fraction $\gamma$ of the capacity region, i.e.,

$$\begin{aligned} \gamma^* := \sup\{\gamma \mid\ & \text{the system is stable under all offered} \\ & \text{load vectors } \vec{\lambda} \text{ such that } \vec{\lambda} \preceq \gamma\vec{\phi} \\ & \text{for some } \vec{\phi} \in Co(\mathcal{M}_E)\}. \end{aligned} \tag{2}$$

## III. An Equivalent Characterization of the Efficiency Ratio of *GMM*

In this section, we derive an equivalent characterization of the efficiency ratio of *GMM* under arbitrary network topologies. We first recall the following definition of *local pooling* from [17]:

*Definition 3:* Given a network graph $G(V, E)$, a set of links $L \subset E$ satisfies *local pooling*, if there exists a nonzero $\vec{\alpha} \in \mathbb{R}_+^{|L|}$ such that $\vec{\alpha}^T \vec{\phi}$ is a positive constant for all $\vec{\phi} \in Co(\mathcal{M}_L)$. The graph $G(V, E)$ satisfies *local pooling* if every $L \subset E$ satisfies local pooling.

An example of graphs that satisfy local pooling is the triangular network topology with three nodes and three links. In this graph, we have three maximal matchings; $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$, where $[\cdot]$ represents a column vector. For any convex combination $\vec{\phi}$ of these three vectors, we have $\vec{\alpha}^T\vec{\phi} = 1$ with $\vec{\alpha} = [1,1,1]$.

Note that if a set of links $L$ satisfies local pooling, no vector in $Co(\mathcal{M}_L)$ strictly dominates another vector in $Co(\mathcal{M}_L)$[1]. Dimakis and Walrand [17] have shown that if a network graph satisfies local pooling, *GMM* achieves the full capacity region.

In this paper, we are interested in arbitrary network topologies that may not satisfy local pooling. We now generalize the notion of local pooling to that of the local-pooling factor.

*Definition 4:* A set of links $L$ satisfies *$\sigma$-local pooling*, if $\sigma\vec{\mu} \not\succeq \vec{\nu}$ for all $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)$. In other words, for all $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)$, there must exist some $k \in L$ such that $\sigma\mu_k < \nu_k$.

Note that if a graph, e.g., the triangular network topology, satisfies local pooling, then it must satisfy $\sigma$-local pooling for any $\sigma < 1$. We can prove this by contradiction. Suppose that there exist two convex combinations $\vec{\phi}_1, \vec{\phi}_2$ and $\sigma < 1$ such that $\sigma\vec{\phi}_1 - \vec{\phi}_2 \succeq \vec{0}$. Since the graph satisfies local pooling, there exists $\vec{\alpha}$ such that $\vec{\alpha}^T\vec{\phi}_1 = \vec{\alpha}^T\vec{\phi}_2 > 0$. Multiplying $\vec{\alpha}$ to both sides of $\sigma\vec{\phi}_1 - \vec{\phi}_2 \succeq \vec{0}$, we obtain $\sigma - 1 \geq 0$, which contradicts the assumption.

*Definition 5:* The *local-pooling factor* of a graph $G(V,E)$ is the supremum of all $\sigma$ such that every subset $L \in E$ satisfies $\sigma$-local pooling. In other words,

$$\begin{aligned} \sigma^* :=& \sup\{\sigma |\ \sigma\vec{\mu} \not\succeq \vec{\nu} \text{ for all } L \text{ and all } \vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)\} \\ =& \inf\{\sigma |\ \sigma\vec{\mu} \succeq \vec{\nu} \text{ for some } L \text{ and some } \vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)\}. \end{aligned} \tag{3}$$

By definition, if the local-pooling factor of a graph is $\sigma^*$, then every subset $L \subset E$ must satisfy $\sigma^*$-local pooling. Note that Definition 3 of local pooling corresponds to $\sigma^* = 1$. The results of [17] imply that if the local-pooling factor of the graph is 1, then the efficiency ratio of *GMM* will be 1. We next generalize this result to the case when $\sigma^* < 1$. We start with two lemmas.

*Lemma 6:* If the local-pooling factor of a graph $G(V,E)$ is $\sigma^*$, then the efficiency ratio $\gamma^*$ of *GMM* under this network topology is no smaller than $\sigma^*$.

*Proof:* We need to show that for any offered load $\vec{\lambda}$ strictly within $\sigma^*\Lambda$, the network is stable under *GMM*. We prove stability by finding a Lyapunov function with negative drift for the fluid limit model of

[1] We can prove this by contradiction. Suppose that there exist $\vec{\phi}_1, \vec{\phi}_2 \in Co(\mathcal{M}_L)$ such that $\vec{\phi}_1 \succ \vec{\phi}_2$. Multiplying $\vec{\alpha}$ to both sides, we obtain $\vec{\alpha}^T\vec{\phi}_1 > \vec{\alpha}^T\vec{\phi}_2$, which contradicts the assumption.

the system.

We first define the fluid limit model of the system as in [4], [23]. Let $A_l(t)$ denote the number of packets that arrive at link $l$ at time slot $t$ and let $S_l(t)$ denote the number of packets that link $l$ can serve at time slot $t$. Let $Q_l(t)$ denote the number of packets queued at link $l$ at the beginning of time slot $t$. The queue length then evolves according to the following equation:

$$Q(t+1) = [Q_l(t) + A_l(t) - S_l(t)]^+,$$

where $[\cdot]^+$ denote the projection to non-negative real numbers. We can interpolate the values of $A_l(t)$ and $S_l(t)$ to all non-negative real number $t$ by setting $A_l(t) = A_l(\lfloor t \rfloor)$ and $S_l(t) = S_l(\lfloor t \rfloor)$, where $\lfloor t \rfloor$ denotes the largest integer smaller than or equal to $t$. We also interpolate the values of $Q_l(t)$ by linear interpolation between $\lfloor t \rfloor$ and $\lfloor t \rfloor + 1$. Then, using the techniques of Theorem 4.1 of [23], we can show that, for almost all sample paths and for all positive sequence $x_n \to \infty$, there exists a subsequence $x_{n_j}$ with $x_{n_j} \to \infty$ such that the following convergence holds uniformly over compact intervals of time $t$: For all $l \in E$,

$$\begin{aligned} \frac{1}{x_{n_j}} \int_0^{x_{n_j} t} A_l(s) ds &\to \lambda_l t, \\ \frac{1}{x_{n_j}} \int_0^{x_{n_j} t} S_l(s) ds &\to \int_0^t \pi_l(s) ds, \\ \frac{1}{x_{n_j}} Q_l(x_{n_j} t) &\to q_l(t). \end{aligned}$$

Moreover, for all $l \in E$, the limits $q_l(t)$ and $\pi_l(t)$ satisfy

$$\frac{d}{dt} q_l(t) = \begin{cases} \lambda_l - \pi_l(t), & \text{if } \lambda_l - \pi_l(t) \geq 0, \text{ or } q_l(t) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and $\pi_l(t)$ must satisfy the requirement of *GMM*. Any such limit $[\vec{q}(t), \vec{\pi}(t)]$ is called a ***fluid limit*** of the system.

We now use the idea from [17] and show that for any offered load $\lambda$ strictly within $\sigma^* \Lambda$, the largest queue length of the fluid limit model always decreases under *GMM*. Note that $q_l(t)$ is absolutely continuous, and hence its derivative exists almost everywhere. Consider those times $t$ when the derivative $\frac{d}{dt} q_l(t)$ exists for all $l \in E$. Let $L_0(t)$ denote the set of links with the longest queue at time $t$, i.e.,

$$L_0(t) := \left\{ l \in E \;\middle|\; q_l(t) = \max_{k \in E} q_k(t) \right\}.$$

Let $L(t)$ denote the set of links with the largest derivative of the queue length among the links in $L_0(t)$,

$$L(t) := \left\{ l \in L_0(t) \;\middle|\; \frac{d}{dt}q_l(t) = \max_{k \in L_0(t)} \frac{d}{dt}q_k(t) \right\}.$$

Then, the links in $L(t)$ will have the longest queue within a small time interval $(t, t+\delta]$. Hence, *GMM* will try to serve these links first. Let the service rate vector of *GMM* be $\vec{\pi}(t)$. Note that $\vec{\pi}(t)$ must be a convex combination of the maximal matchings on $L(t)$, i.e., $\vec{\pi}(t) \in Co(\mathcal{M}_{L(t)})$. Since the local-pooling factor is $\sigma^*$, and $\vec{\lambda}$ falls strictly within $\sigma^*\Lambda$, there must exist a $k \in L(t)$ such that $\lambda_k < \pi_k(t)$.

Define

$$\epsilon_* := \inf_{\vec{\pi} \in Co(\mathcal{M}_L), L \subset E} \left\{ \max_{k \in L} (\pi_k - \lambda_k) \right\}. \tag{4}$$

Note that $\epsilon_* > 0$ since the local pooling factor is $\sigma^*$ and $\vec{\lambda}$ falls strictly within $\sigma^*\Lambda$. Hence, by the earlier argument, there exists a $k \in L(t)$ such that $\lambda_k - \pi_k(t) \le -\epsilon_*$. This implies that $\frac{d}{dt}q_l(t) = \frac{d}{dt}q_k(t) \le -\epsilon_*$ for all $l \in L(t)$, i.e., the largest queue length must decrease in the time interval $(t, t+\delta]$.

Therefore, we can pick the Lyapunov function as $V(t) := \max_{l \in E} q_l(t)$. We have, if $V(t) > 0$,

$$\frac{D^+}{dt^+}V(t) \le \max_{l \in L_0(t)} \frac{d}{dt}q_l(t) = \left. \frac{d}{dt}q_k(t) \right|_{k \in L(t)} \le -\epsilon_*,$$

where $\frac{D^+}{dt^+}V(t) = \lim_{\delta \downarrow 0} \frac{V(t+\delta) - V(t)}{\delta}$. Since this is true for almost every $t$, it implies that the fluid limit model of the system is stable. By Theorem 4.2 of [23], the original system is also stable. ■

Lemma 6 shows that the efficiency ratio of *GMM* under an arbitrary network graph is no smaller than the local-pooling factor, i.e., $\gamma^* \ge \sigma^*$.

The next lemma shows that $\gamma^* \le \sigma^*$.

*Lemma 7:* If there exist a subset of links $L \subset E$, a positive number $\sigma$, and two vectors $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)$ such that $\sigma\vec{\mu} \succeq \vec{\nu}$, then, for arbitrarily small $\epsilon > 0$, there exists a traffic pattern with offered load $\vec{\nu} + \epsilon\vec{e}_L$ such that the system is unstable under *GMM*, where $\vec{e}_L$ is a vector with $e_l = 1$ for $l \in L$ and $e_l = 0$ for $l \notin L$.

*Remark:* Since $\vec{\nu} \in \sigma\Lambda$, Lemma 7 implies that the efficiency ratio of *GMM* under this network topology is no greater than the local-pooling factor, i.e., $\gamma^* \le \sigma^*$.

*Proof:* We will construct a traffic pattern with offered load $\vec{\nu} + \epsilon\vec{e}_L$ based on $\vec{\nu}$, and show that under this traffic pattern, the queue length will increase to infinity under *GMM*.

Let $\aleph$ denote the number of all maximal matchings $M_i$ on $L$. Since $\vec{\nu}$ is a convex combination of these maximal matchings, it can be written as

$$\vec{\nu} = \sum_{i=0}^{\aleph-1} w_i M_i,$$
$$\text{where} \quad w_i \geq 0 \quad \text{for all} \quad 0 \leq i \leq \aleph - 1, \quad \text{and} \quad \sum_{i=0}^{\aleph-1} w_i = 1. \tag{5}$$

For each $w_i$, we can find a rational number $v_i$ such that $|w_i - v_i| \leq \frac{\delta}{\aleph}$ for any $\delta > 0$ and thus,

$$\sum_{i=0}^{\aleph-1} |w_i - v_i| \leq \delta. \tag{6}$$

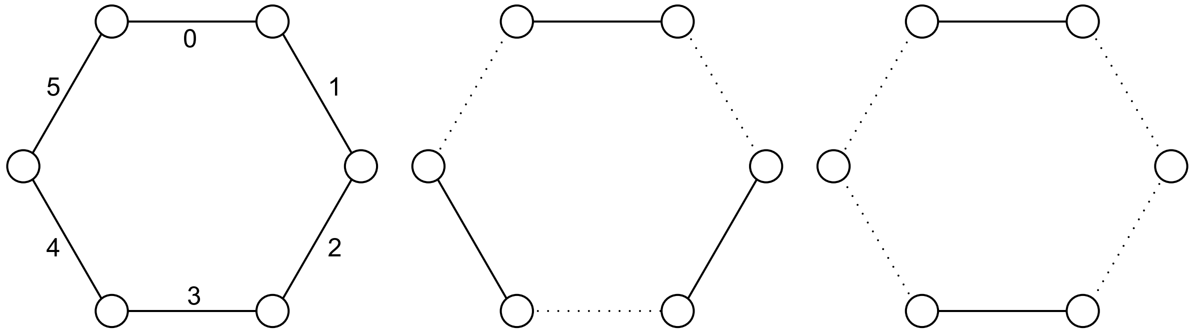
Using these rational numbers, we define a new vector $\vec{\nu}' := \sum_{i=0}^{\aleph-1} v_i M_i$.

We now construct a traffic pattern with offered load $\vec{\lambda} = \vec{\nu}' + \epsilon \vec{e}_L$ such that the system is unstable under *GMM*. We assume that packets arrive to a link *before* a time slot and the queue of all links in $L$ is empty at the beginning.

Let $T$ denote the smallest number such that, for all $i$, $v_i T$ is an integer. Let $t_i = v_i T$. Assume without loss of generality that $t_{\aleph-1} \geq 1$.

- For the first $t_0$ time slots, we apply one packet every time slot to links included in $M_0$. Then, each time slot, *GMM* serves all the links included in $M_0$ since they have the longest queues and do not interfere with each other. Hence, at the end of $t_0$ time slots, all queues in $L$ will have the same queue length.
- Similarly, for each $i = 1, 2, \ldots, \aleph - 2$, we apply one packet every time slot for $t_i$ time slots to links included in $M_i$. For the same reason as above, in each of $t_i$ time slots, *GMM* serves all links in $M_i$ since they have the longest queues. At the end of each of the $t_i$ time slots, all queues in $L$ will have the same queue length.
- For $i = \aleph - 1$, we apply one packet every time slot for $t_{\aleph-1} - 1$ time slots to links included in $M_{\aleph-1}$. Then in the next time slotm *with probability* $1 - \epsilon'$, we apply one packet to links included in $M_{\aleph-1}$, and *with probability* $\epsilon'$, apply two packets to links included in $M_{\aleph-1}$ and one packet to all other links in $L$. Note that *GMM* still serves the links in $M_{\aleph-1}$ in each of the $t_{\aleph-1}$ time slots. Hence, at the end of $t_{\aleph-1}$ time slots, all queues $L$ will still have the same queue length. However, with probability $\epsilon'$, the queue length increases by 1.

The above pattern then repeats itself so that the same pattern of arrivals occurs every $\sum_{i=0}^{\aleph-1} t_i$ time slots.




(a) Topology (b) Maximal matching $M_0$ (c) Maximal matching $M_2$

Fig. 1. The 6-link cycle network and the instances of maximal matching. The solid lines in (b) and (c) are the active links.

Note that the average arrival rate of this traffic pattern is $(\sum_{i=0}^{\aleph-1} t_i M_i + \epsilon' \vec{e}_L)/(\sum_{i=0}^{\aleph-1} t_i)$ and the queue length increases by 1 with probability $\epsilon'$ every $\sum_{i=0}^{\aleph-1} t_i$ time slots. Letting $\epsilon = \frac{\epsilon'}{\sum t_i}$, then $(\sum_{i=0}^{\aleph-1} t_i M_i + \epsilon' \vec{e}_L)/(\sum_{i=0}^{\aleph-1} t_i) = \vec{\nu}' + \epsilon \vec{e}_L$. Hence, we have shown that the system with offered load $\vec{\nu}' + \epsilon \vec{e}_L$ is unstable under *GMM*. From (6), we can choose $\delta$ such that the difference between $\vec{\nu}'$ and $\vec{\nu}$ arbitrary small. Hence, the system with $\vec{\nu} + \epsilon \vec{e}_L$ is unstable under *GMM*. ■

Note that the key to the proof is to construct a traffic pattern such that (i) it keeps all queues in $L$ of the same length, and (ii) it injects packets according to the maximal matchings that form the vector $\vec{\nu}$ so that these maximal matchings will be picked by *GMM*.

*Example:* The following example illustrates how such a traffic pattern can be constructed in the 6-link cycle network shown in Fig. 1. We number all links clockwise from 0 to 5. All possible maximal matchings under this network graph are listed below.

- $M_0 = [1,0,1,0,1,0],\ M_1 = [0,1,0,1,0,1]$,
- $M_2 = [1,0,0,1,0,0],\ M_3 = [0,0,1,0,0,1[,\ M_4 = [0,1,0,0,1,0]$.

Note that the number of links included in a maximal matching is three for $M_0$ and $M_1$, and is two for $M_2$, $M_3$, and $M_4$. Figs. 1(b) and 1(c) show the two instances of the maximal matchings, i.e., $M_0$ and $M_2$. Note that if we choose two vectors $\vec{\mu}, \vec{\nu}$ from the convex set of maximal matchings $Co(\{M_i\})$ as

$$\vec{\mu} = \frac{1}{2}M_0 + \frac{1}{2}M_1 = \left[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]$$
$$\vec{\nu} = \frac{1}{3}M_2 + \frac{1}{3}M_3 + \frac{1}{3}M_4 = \left[\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right].$$

then $\frac{2}{3}\vec{\mu} \succeq \vec{\nu}$.

We now construct a traffic pattern with offered load $\vec{\lambda} = \vec{\nu} + \frac{\epsilon}{3}\vec{e}$ such that the system is unstable under *GMM*, where $\vec{e} = [1, 1, 1, 1, 1, 1]$ and $\epsilon$ is a small positive number. Assume that all queues in the system are of the same length at time 0.

1) *1st time slot:* One packet is applied to links 0 and 3. Since *GMM* gives priority to links with a longer queue, it will serve links 0 and 3. Therefore, at the end of time slot 1, all queues will still have the same length.
2) *2nd time slot:* One packet is applied to links 1 and 4. For the same reason as above, *GMM* will serve links 1 and 4, and all queues will still have the same length at the end of time slot 2.
3) *3rd time slot:* With probability $1 - \epsilon$, one packet is applied to links 2 and 5. With probability $\epsilon$, two packets are applied to links 2 and 5, and one packet is applied to all other links. In both cases, links 2 and 5 have the longest queue and will be served by *GMM*. At the end of time slot 3, all queues still have the same length. However, with probability $\epsilon$, the queue length increases by 1.

The pattern then repeats itself.

Over all links, the arrival rate is $\frac{1}{3} + \frac{\epsilon}{3}$ and the queue length increases by 1 with probability $\epsilon$ every three time slots. Hence, the system with offered load $\vec{\nu} + \frac{\epsilon}{3}\vec{e}$ is unstable under *GMM*. However, the optimal *MWM* policy can support the offered load $\vec{\mu} = \frac{3}{2}\vec{\nu}$ in this example. Hence, the efficiency ratio of *GMM* is no greater than $\frac{2}{3}$ in this 6-link cycle network.

From Lemmas 6 and 7, we can conclude that:

*Proposition 8:* The efficiency ratio $\gamma^*$ of *GMM* under a given network topology is equal to its local-pooling factor $\sigma^*$.

This result provides an equivalent characterization of the efficiency ratio of *GMM* through the topological properties (i.e., the local-pooling factor) of the given graph. Unfortunately, it can still be quite difficult to compute the local-pooling factor for an arbitrary network graph. We next present some preliminary results for estimating the local-pooling factors.

## IV. Estimates of the Local-Pooling Factor for Arbitrary Network Graphs

In this section, we would like to answer the following questions: (i) how do we estimate the local-pooling factor of a given graph? and (ii) what types of graphs will have low local-pooling factors? We now argue that both questions are intimately related to the characterization of the possible sets of unstable links. Note that in order to claim $\sigma^* \leq \sigma$, we must find a subset of links $L$, and two vectors $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)$ such that $\sigma\vec{\mu} \succeq \vec{\nu}$. In fact, in the proof of Lemma 7, we show that for any $\epsilon > 0$, there exists a traffic pattern with offered load $\vec{\nu}+\epsilon\vec{e}_L$ such that the queues of all links in $L$ increase to infinity together under *GMM*. Hence, a starting point to search for $\vec{\mu}$ and $\vec{\nu}$ would be to find out which subset of links $L$ could have queue length increasing to infinity together under *GMM* at such an offered load $\vec{\lambda} \in \Lambda = \vec{\nu} + \epsilon\vec{e}_L$ and under such a traffic pattern. In this paper, we provide some preliminary results along this direction.

To avoid confusion, we let $Y$ denote the set of links in $E$ whose queue lengths increase to infinity together under *GMM* at offered load $\vec{\lambda} = \vec{\nu} + \epsilon\vec{e}_Y$, where $\vec{\nu} \in Co(\mathcal{M}_Y)$. By constructing the traffic pattern as in (1)-(3) earlier in the proof of Lemma 7, we have $Q_l(t) = 0$ for all $l \notin Y$, $Q_l(t) = Q(t)$ for all $l \in Y$, and $Q(t_k) \to \infty$ for a sequence $\{t_1, t_2, \ldots, \infty\}$. We refer to the links in $Y$ as the *unstable links*. Let $X$ denote the set of nodes connected to any of the links in $Y$. We call the graph $U(X,Y)$ an *unstable subgraph* of $G(V,E)$, We next define the notion of an *isolated unstable* link and an *open unstable* link in the unstable subgraph $U(X,Y)$.

*Definition 9:* A link $l \in Y$ connecting two nodes $n_1$ and $n_2$ is an *isolated unstable* link if both $n_1$ and $n_2$ are of degree 1 in the unstable subgraph $U(X,Y)$.

*Definition 10:* A link $l \in Y$ connecting two nodes $n_1$ and $n_2$ is an *open unstable* link if either $n_1$ or $n_2$ is of degree 1 in the unstable subgraph $U(X,Y)$.

We have the following two results.

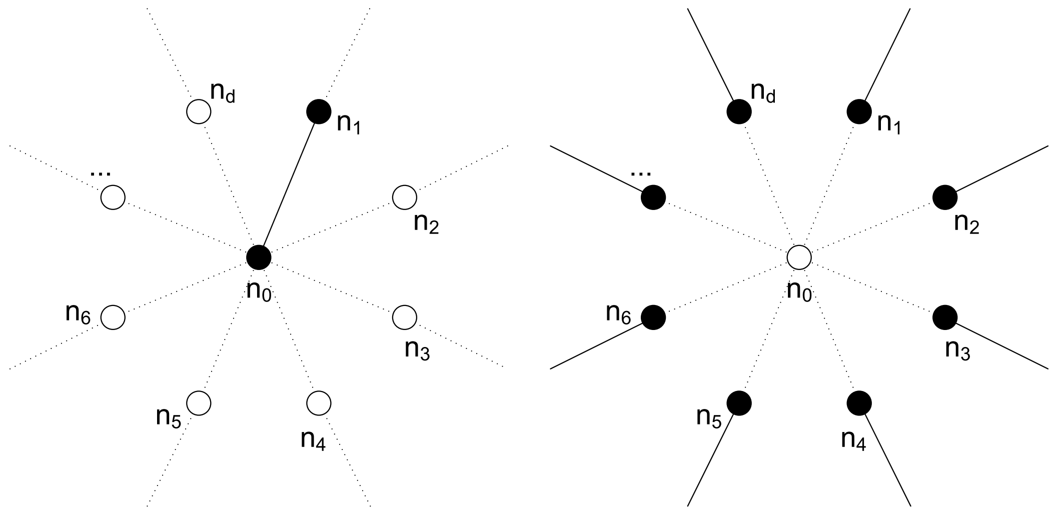
*Lemma 11:* If $\vec{\lambda} = \vec{\nu} + \epsilon\vec{e}_Y$ is strictly within $\Lambda$, then there is no isolated unstable link in $Y$ under *GMM*.

*Proof:* Suppose that $Y$ includes an isolated link $l$. By assumption, link $l$ has no neighboring links in $Y$ and should be included in all maximal matchings on $Y$. As a result, link $l$ will be selected at all time slots by *GMM*. Since $\lambda_l < 1$, the queue length of link $l$ cannot increase to infinity. This contradicts the assumption that link $l$ is unstable. ■

*Lemma 12:* If $\vec{\lambda} = \vec{\nu} + \epsilon\vec{e}_Y$ is strictly within $\Lambda$, there is no open unstable link in $Y$ under *GMM*.

*Proof:* Suppose that $Y$ includes an open unstable link $l_0 = (n_1, n_2)$. Without loss of generality, assume that node $n_1$ is shared by other unstable links $\{l_1, l_2, \ldots, l_i\} \subset Y$, and node $n_2$ is of degree 1 in $Y$.

Note that every maximal matching on $Y$ should include at least one of the links $l_0, l_1, \ldots, l_i$; because, if none of $l_1, \ldots, l_i$ is included, link $l_0$ should then be included in order for the matching to be maximal in $Y$. Hence, under *GMM*, the sum of the service rates over all of these links is 1 at all time slots. Recall that all queues of links $l_0, l_1, \ldots, l_i$ are of the same length at $t = t_1, t_2, \ldots$. Since $\sum_{k\in\{l_0,l_1,\ldots,l_i\}} \lambda_k < 1$,




(a) Node $n_0$ is black

(b) Node $n_0$ is white

Fig. 2. Maximal matchings on an unstable network with a degree-$d$ node $n_0$.

these links cannot be unstable. This contradicts the assumption that the queues of these links increase to infinity together. ■

The above two lemmas imply that any link in $Y$ must belong to a cycle formed by links in $Y$. Note that it immediately implies the result that *GMM* achieves the full capacity region in tree networks [17], [18].

In the following lemma, we characterize the property of the unstable subgraph when the arrival rate $\vec{\lambda}$ is within $\gamma\Lambda$.

*Lemma 13:* Suppose $\gamma \in (1/2, 1]$ and $\vec{\lambda} = \vec{\nu} + \epsilon\vec{e}_Y$ is strictly within $\gamma\Lambda$, then the degree of every node $v \in X$ in the unstable subgraph $U(X, Y)$ must be larger than $\frac{\gamma}{2\gamma-1}$.

*Proof:* We consider a node $n_0 \in X$ of degree $d$ (in $X$) with neighbors $\{n_1, n_2, \ldots, n_d\} \subset X$. Let $l_i$ denote link $(n_0, n_i)$ and let $\mathcal{L}_i$ denote the set of unstable links connected to $n_i$, excluding $l_i$, i.e., $\mathcal{L}_i = Y \cap \mathcal{E}(n_i) \backslash \{l_i\}$, where $\mathcal{E}(n_i) \subset E$ is the set of links that are connected to node $n_i$.

Observe that all maximal matchings on $Y$ must fall into one of the following two cases:

1) A maximal matching on $Y$ includes a link $l_i$. In this case, we say that node $n_0$ is black (see Fig. 2(a)).
2) A maximal matching on $Y$ includes a link from each $\mathcal{L}_i$. In this case, we say that node $n_0$ is white (see Fig. 2(b)).

We first show that the fraction of time that $n_0$ is black (the first case) is no more than $\gamma$. Let $\lambda_n := \sum_{l\in\mathcal{E}(n)} \lambda_l$ denote the arrival rate at node $n$, and let $D_n := \sum_{l\in\mathcal{E}(n)} D_l$ denote the departure rate at node $n$, where $D_l$ is the departure rate at link $l$. Note that the optimal capacity region $\Lambda$ is bounded by

$$\Lambda \subset \Psi := \left\{ \vec{\lambda} \;\middle|\; \sum_{l\in\mathcal{E}(n)} \lambda_l \le 1, \text{ for all } n \in E \right\}.$$

By assumption $\vec{\lambda} \in \gamma\Lambda$, we have

$$\sum_{l\in\mathcal{E}(n_0)\cap Y} \lambda_l \le \sum_{l\in\mathcal{E}(n_0)} \lambda_l = \lambda_{n_0} \le \gamma. \tag{7}$$

If the fraction of time that $n_0$ is black is greater than $\gamma$, then the arrival rate at node $n_0$ will be smaller than the service rate at $n_0$, which implies that the queues at the links incident to node $n_0$ cannot increase to infinity together. This contradicts our assumption.

We next count the total service rates over all nodes $n_0, n_1, \ldots, n_d$. Let $\beta$ denote the fraction of time that node $n_0$ is black, $0 \le \beta \le \gamma$. If node $n_0$ is black, then at least two nodes (one is $n_0$) are served. If node $n_0$ is white, then nodes $\{n_1, n_2, \ldots, n_d\}$ are served. Hence, we have

$$\sum_{k=0}^{d} D_{n_k} \ge 2\beta + d(1-\beta) \ge 2\gamma + d(1-\gamma). \tag{8}$$

In the last inequality, we have used $0 \le \beta \le \gamma$ and $d \ge 2$ (by Lemma 12).

Using the assumption that $\vec{\lambda}$ falls strictly in $\gamma\Psi$, we have

$$\sum_{k=0}^{d} \lambda_{n_k} < \gamma(d+1). \tag{9}$$

We must have

$$\sum_{k=0}^{d} D_{n_k} \le \sum_{k=0}^{d} \lambda_{n_k}, \tag{10}$$

since, otherwise, the queue lengths of these links cannot increase to infinity together. Combining (8), (9), and (10), we obtain

$$d > \frac{\gamma}{2\gamma - 1}. \tag{11}$$

■

The above lemma immediately implies the second main result of the paper.

*Proposition 14:* For a given network graph $G(V, E)$ where the largest node degree is $d^*$, the efficiency ratio $\gamma^*$ of *GMM* must be no smaller than $\frac{d^*}{2d^*-1}$.

*Proof:* Suppose that the efficiency ratio is smaller than $\frac{d^*}{2d^*-1}$. Then, according to Proposition 8, we have $\sigma^* < \frac{d^*}{2d^*-1}$. Hence, from Definition 5, there must exist a subset $L \subset E$ and $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)$ such that $\sigma\vec{\mu} \geq \vec{\nu}$ for some $\sigma < \frac{d^*}{2d^*-1}$. Using Lemma 7, there exists a traffic pattern with $\vec{\lambda} = \vec{\nu} + \epsilon\vec{e}_L$, such that the queue lengths of links in $L$ increase to infinity together. By choosing $\epsilon$ small, we can have $\vec{\lambda}$ fall strictly in $\frac{d^*}{2d^*-1}\Lambda$. Then, using Lemma 13, the degree of every node in the unstable graph must be larger than $d^*$. This contradicts the assumption that the largest node-degree is $d^*$. ■

According to Proposition 14, in order to find network topologies where the efficiency ratio of *GMM* is low, we must look at those graphs where the maximum node-degree is high. We have been able to find such graphs where the bound in Proposition 14 is tight with $d^* = 2$ and $d^* = 3$.

### A. *An example network scenario with $d^* = 2$ and $\gamma^* = \frac{2}{3}$*

We consider graphs with degree two. If the graph is a line, then *GMM* achieves the full capacity region by Lemma 12. Let us instead consider th case when the graph forms a cycle. In the proof of Lemma 7, we show an example of a 6-link cycle network, which has $\gamma^* \leq \frac{2}{3}$. Since this graph has a maximum node-degree of two, Lemma 13 implies that $\gamma^* \geq \frac{2}{3}$. Therefore, *GMM* has an efficiency ratio $\gamma^* = \frac{2}{3}$ in the 6-link cycle network. To the best of our knowledge, *this is the first result that provides the* exact *efficiency ratio for a network graph where GMM cannot achieve the full capacity region.*

### B. *An example network scenario with $d^* = 3$ and $\gamma^* = \frac{3}{5}$*

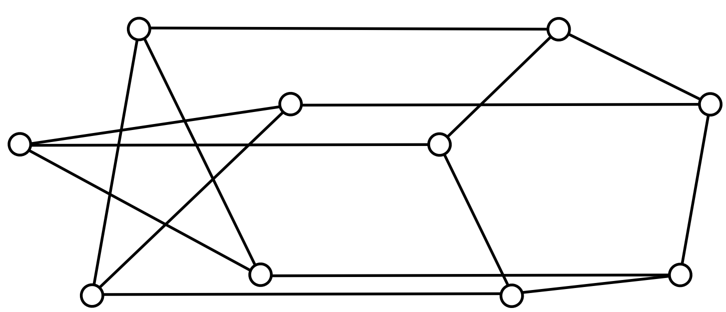
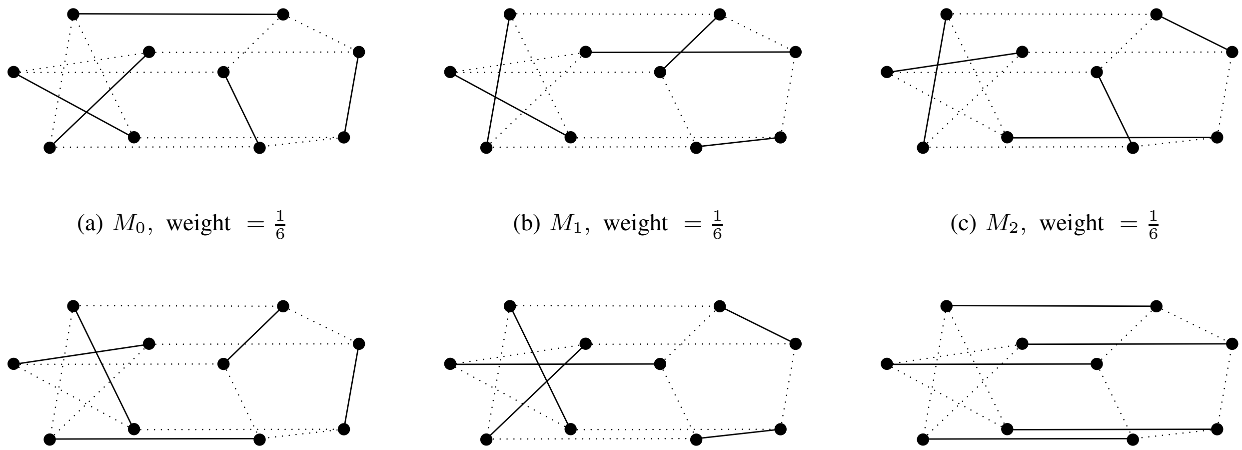


Fig. 3. Star-pentagon Topology

We consider the graph with node-degree three as shown in Fig. 3. We now find two vectors $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_E)$ such that $\frac{3}{5}\vec{\mu} = \vec{\nu}$. Fig. 4 shows six maximal matchings and their corresponding weights.



(a) $M_0$, weight $= \frac{1}{6}$ (b) $M_1$, weight $= \frac{1}{6}$ (c) $M_2$, weight $= \frac{1}{6}$

(d) $M_3$, weight $= \frac{1}{6}$ (e) $M_4$, weight $= \frac{1}{6}$ (f) $M_5$, weight $= \frac{1}{6}$

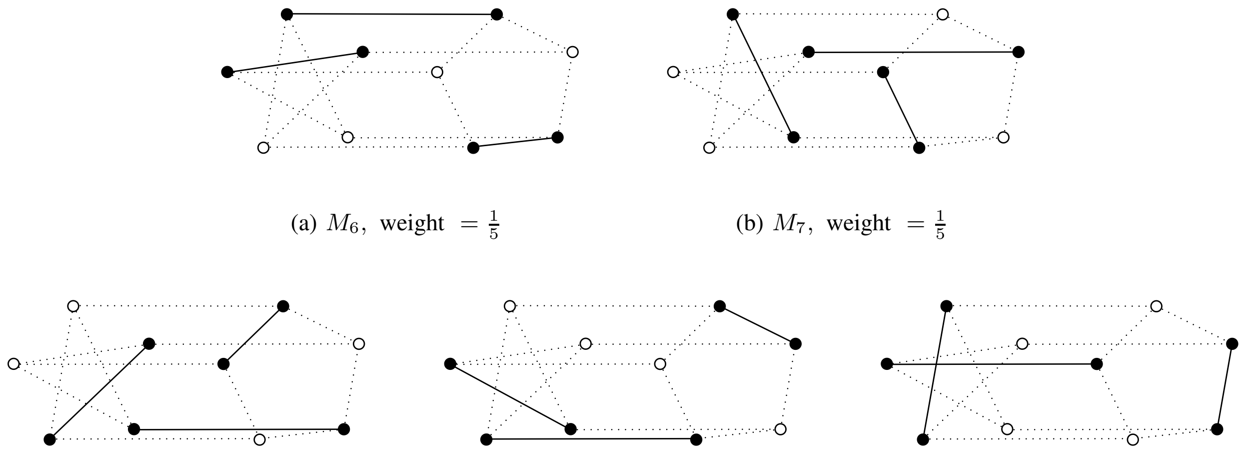
Fig. 4. Maximal matchings for constructing $\vec{\mu}$.

The solid lines indicate active links. We choose vector $\vec{\mu}$ as a combination of these matchings, i.e., $\vec{\mu} = \sum_{i=0}^{5} \left(\frac{1}{6} M_i\right)$. Fig. 5 illustrates another set of maximal matchings. We choose $\vec{\nu}$ using these matchings as $\vec{\nu} = \sum_{j=6}^{10} \left(\frac{1}{5} M_j\right)$. Note that $\mu_l = \frac{1}{3}$ and $\nu_l = \frac{1}{5}$ for all links $l$.

Since $\frac{3}{5}\vec{\mu} = \vec{\nu}$, the local-pooling factor $\sigma^*$ cannot be greater than $3/5$, which implies that the efficiency ratio of *GMM* is no greater than $3/5$. However, since the node degree is 3, Proposition 14 implies that the efficiency ratio is no smaller than $3/5$. Hence, the efficiency ratio is exactly $3/5$.

## V. Conclusion

In this paper, we have provided new analytical results on the achievable performance of *GMM* for a large class of network topologies. We derive our results via a topological approach that extends the recently developed notion of local pooling to a more general topological notion called $\sigma$-local pooling, and a corresponding notion called local-pooling factor. We show that for a given graph, the efficiency ratio of *GMM* is equal to its local-pooling factor. Thus, we are able to focus on the topological property of graphs to obtain the achievable performance of *GMM*. However, it turns out that estimating the local-pooling factor is non-trivial, and may require high complexity for arbitrary network topologies. Nonetheless, by studying the properties of unstable networks, we are able to provide preliminary results on estimating the local-pooling factor of arbitrary network topologies. In particular, we show that the local-pooling factor (and hence the efficiency ratio $\gamma^*$ of *GMM*) of a graph with maximum node degree $d^*$ is no smaller than



(a) $M_6$, weight $= \frac{1}{5}$ (b) $M_7$, weight $= \frac{1}{5}$

(c) $M_8$, weight $= \frac{1}{5}$ (d) $M_9$, weight $= \frac{1}{5}$ (e) $M_{10}$, weight $= \frac{1}{5}$

Fig. 5. Maximal matchings for constructing $\vec{\nu}$.

$d^*/(2d^* - 1)$. The tightness of the bound is demonstrated through the 6-link cycle and the Star-pentagon topologies, where $d^* = 2$ and $d^* = 3$, respectively.

There remain many interesting open problems in these directions. For example, further research on the topological properties of graphs could result in a better estimate of the performance limits. We also expect that different interference models will affect the capacity region of *GMM*. While our results on the relationship between the performance of *GMM* and the local-pooling factor remain the same for a more general class of interference models, more work needs to be done to evaluate the local-pooling factor for general interference models. Finally, the authors of [17] show that, if the arrivals satisfy certain randomness property, *GMM* may achieve the full capacity region even if the network graph does not satisfy local pooling. It would be interesting to study whether the results in this paper can be improved under similar assumptions.

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