Stable Real-time Pricing and Scheduling for Serving Opportunistic Users with Deferrable Loads

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Abstract— **In this paper, we address real-time pricing and control of opportunistic consumers with deferrable demands that are motivated by the envisioned smart electrical grid. In the smart grid, demand-side flexibilities from deferrable loads enable consumers to respond to real-time electricity prices for their own economic benefit. However, the aggregate load created by many such economically-driven consumers can be highly time-varying, which can cause significant fluctuations in both the electricity demand and the real-time price, and ultimately incur additional generation and capacity costs for the suppliers. In this paper, we propose a distributed pricing and load scheduling algorithm that alleviates such undesirable fluctuations and high volatility. In particular, we formulate the pricing and scheduling problem as an optimization problem with proximal terms that incur penalty for rapid *consumption changes over time*. Through a continuous-time approximation, we show how the overall system evolves towards the optimal operating regime under the proposed algorithm. In the limiting operating regime, even though the individual consumption of each consumer may still exhibit oscillatory behavior, the aggregate load and the real-time price at any time stay arbitrarily close to a stable operating point. Our design also enables third-party intermediaries to adjust the change of load penalty to balance the payments made by the consumers to the suppliers, thereby achieving market clearance.**

I. Introduction

This work aims to develop simple real-time pricing schemes for serving economically-driven consumers with stochastically arriving and flexible demands. While the resulting insights can apply in greater generality to other systems (most notably smart data networks), we focus on the pricing for the electricity power grid. The future smart grid is envisioned to contain various forms of flexibility in the service demands, e.g., deferrable service times as in operating a dishwasher overnight, controllable power levels as in lighting or heating/cooling systems, etc. One of the visions is that the flexibility of demand from these users could dynamically be controlled to reduce the peak consumption, and therefore increase the efficiency of the overall system. However, users are self-interested agents and will allow such flexibilities to be used only if they can get economical benefits. Therefore, it is necessary to design intelligent pricing schemes that not only benefit the grid, but also provide the monetary incentives for the users to alter their load.

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However, such self-interested users will likely utilize their flexibilities to their advantage by opportunistically responding to the changing prices, which may potentially lead to highly volatile behavior. For example, the authors of [1] study the optimal user behavior when the prices are assumed to be exogenously generated, and establish that an asymptotically optimal policy possesses an aggressive threshold-based behavior. Similar observations have also been made in [2] under different flexibility and cost structures, and in non-asymptotic regimes. The main insight from these works, which captures a general and perhaps obvious consumer characteristic, is that, within the limits of the users' flexibilities, the users will tend to apply load when the price is low. While this threshold-type behavior is aligned with the users' own economic interests, it raises immediate concerns that, when the system is serving a large number of such aggressive users, whether the overall system can still be stable.

Such risks of instability have been reported in related work. In fact, if the load is flexible but cannot be shifted in time, [3] shows that if the elasticity of the user demand is large, the system under an ex-ante pricing scheme can indeed become unstable. If the load can be shifted in time, intuitively the elasticity will likely increase further. Hence, this type of instability issues could become a very critical problem for such systems.

The work by [4] attempts to address this problem by introducing ex-ante prices with ex-post adjustments. However, under this scheme the users are forced to predict the future impact of their collective decisions on the system, which is a highly non-trivial task. Another line of work [5] proposes auction mechanisms for demand side management. However, such approaches require both the demand and the users' utility functions to be known in advance, which may be impractical as well.

With this motivation, our goal in this work is to design simple pricing schemes that are both globally stable and also align with the users' economic interest. Our proposed pricing schemes have a similar flavor as ex-ante scheme in the sense that the price is announced before hand. Hence, the users do not need to perform complicated prediction operations. Instead, the key idea of the new proposed scheme is to penalize the changes of load across time, therefore effectively eliminating the extreme-oscillation in the aggregate opportunistic load.

After the description of the system model and problem formulation in Section II-A, our investigations start in Section III by two benchmark pricing strategies that exhibit the volatility of traditional pricing mechanisms. In Section IV, we formulate a related static problem with a proximal term, which leads to a novel pricing mechanism where, in addition to the absolute consumption, the *change in consumption* is also priced. In Section IV-A, we propose a continuous-time approximation of the resulting dynamic system, and study its convergence characteristics. Our analysis reveals an interesting limiting behavior of our scheme, whereby individual consumer loads and queue-evolutions follow sinusoid-like trajectories, they do so in such an asynchronous fashion that the aggregate load converges to a stable value. Our numerical results in Section V confirm the findings of the analytical results also under stochastic conditions.

II. System Model and Problem Formulation

We present a model for the stable control of real-time markets where several suppliers or service providers serve a class of consumers who have the flexibility to defer their demand. We will focus on the volatility and stability issues arising due to the opportunistic behavior of flexible consumers under real-time pricing mechanisms. Although we mainly focus on electricity market in this paper, the basic structure and dynamics of our model may also apply to other markets such as wireless data markets.

A. System Model: Electricity Market

We study a real-time electricity market consisting of electricity consumers, suppliers, and a non-profit party called ISO (Independent System Operator) that oversees the efficient and reliable operation of the market. Time is divided into discrete slots, i.e. $t = 0, 1, \ldots$, that represent appropriate intervals over which the control decisions are made and the total electric supply must be set to meet the load. Figure 1 depicts the market participants and their interactions which are presented in detail in the following.

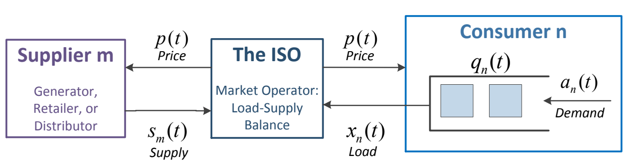

Fig. 1. Market structure showing its participants and their interactions.

1) *ISO (Independent System Operator):* The ISO is a non-profit organization that is responsible for maintaining reliable operation of the market with the objective of achieving a sustainable level of welfare for the market participants [1]. It regulates the load-supply matching and other operational constraints of the electrical grid by controlling the real-time prices which are broadcast to suppliers and consumers. The real-time price per unit of electric power at time slot t is denoted by $p(t)$, and is set and broadcast by the ISO at the beginning of slot t. We assume that the ISO does not have the knowledge of consumers' strategies, valuations of consumption, and their internal states. Hence the demand that will be induced by a given price is uncertain to the ISO.

2) *Suppliers:* There are M electric suppliers such as generator companies, retailers, or load aggregators that are responsible for providing electric power to the grid. The procurement of s watts of power incurs a cost of $C_m(s)$ to the supplier m, where $C_m : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuously differentiable and increasing function of s for each m. We also assume that $\ddot{C}_m(s) > 0$, and hence C_m is strictly convex and $\dot{C}_m$ is invertible. At any time slot, the electric supply provided by the suppliers must meet the load. We assume that the ISO knows the cost function $C_m(\cdot)$ and can use it to adjust the elastic price in the future.

3) *Consumers:* There are N consumers who use electricity. At time slot t, consumer n generates random demand $A_n(t)$. We assume that $A_n(t)$ is independent among users and i.i.d. over time slots for each user. The distribution function of $A_n(t)$ is denoted by F_{A_n} and $\mathbb{E}[A_n(t)] = \lambda_n$ for all t. We further assume that demand arrivals are bounded such that $A_n(t) \in [0, a_{n,max}]$.

The demand is flexible in the sense that it can be deferred by the user before it is actually served as an electric load on the grid. On the other hand, the amount of electric energy consumed by user n at slot t is denoted by $x_n(t) \in [0, x_{max}]$. We emphasize that here we make a clear distinction between the terms *demand* and *load*; at time t user n generates demand $a_n(t)$, which is the realization of random variable $A_n(t)$, and he consumes load $x_n(t)$ based on his own decision. The waiting queue (i.e. backlog) for consumer n's deferrable demand at time t is $q_n(t)$ and its evolution is given by $q_n(t+1) = [q_n(t) + a_n(t) - x_n(t)]^+$. The consumers require their queues to be stable, otherwise the delay experienced by the demand will approach infinity. Our analytical results are mainly based on this stability constraint. On the other hand, in our simulations we will also compare the average delay experienced by the consumers.

Moreover, we assume the consumers are price-taking, thus they determine their loads $x_n(t)$ at the beginning of each time slot t after observing the corresponding price $p(t)$ announced by the ISO (cf. Figure 1). Naturally, the goal of the customer would be to minimize his payment under the above stability constraints. In the paper, we are interested in designing pricing schemes that not only are aligned with the customer's interest, but also optimize the global system performance (to be defined shortly).

The market model induced by the interactions between the ISO, suppliers, and consumers is a closed-loop feedback dynamical system. At the beginning of each time slot, the ISO has to decide on the market price without knowing the amount of load it would induce, because the users' valuations and strategies are not available to the ISO. On the other hand, once the actual load is realized the suppliers have to meet the realized load. Then the ISO decides on the price for the next time slot based on the current realization of the load.

[1]The ISO concept is specific to the electricity market, but the same system model without an ISO applies to other markets such as wireless data. Basically, perfect competition among suppliers would result in an aggregate behavior similar to the ISO's.

B. Problem Formulation

We define $\mathbf{x}(t) \triangleq (x_1(t), \ldots, x_n(t))$ to be the vector of consumer loads at time t and $\mathbf{s}(t)$ to be the vector of electric procurements of suppliers. Let $\{\mathbf{x}(t)\}$ denote the set of $\mathbf{x}(t)$ values for $t = 0, 1, \ldots$. Then, the objective of the ISO is given by the following infinite horizon optimization problem:

$$\min_{\{\mathbf{x}(t)\}\{\mathbf{s}(t)\}} \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{m=1}^{M} \mathbb{E}\left[C_m\left(s_m(t)\right)\right] \tag{1}$$

$$\text{s.t.} \quad \sum_{n=1}^{N} x_n(t) = \sum_{m=1}^{M} s_m(t), \ \forall t = 0, 1, \ldots \tag{2}$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[x_n(t)\right] \geq \lambda_n, \ \forall n \tag{3}$$

In problem (1), the objective function is the limit of the time averaged expected cost of the suppliers for providing electricity. Constraint (3) is a necessary condition for the backlog of each consumer to be stable, and constraint (2) ensures that the electricity provided by the suppliers matches the total consumer load at all time slots. Furthermore, we encapsulate the multiple number of suppliers by defining two cascaded optimization problems in Problem (1). In particular, we modify the objective (1) as $\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[C(s(t))\right]$, and the constraint (2) as $\sum_n x_n(t) = s(t)$, where

$$C(s(t)) \triangleq \min_{\mathbf{s}} \quad \sum_m C_m(s_m(t)) \quad \text{s.t.} \quad \sum_m s_m(t) = s(t). \tag{4}$$

Our goal in this paper is to design pricing schemes that not only minimize the generation cost based on problem (1), but are also aligned with the interest of the consumers to minimize their own payment. In the following section, we first discuss two pricing schemes that have been commonly referred to in the literature and demonstrate that while they are aligned with the consumer's interest, they lead to volatile load and price for the system. Then, In Section IV we will propose a new pricing scheme that eliminates such volatility and instability problems.

Before beginning the analysis we give the following definition for *volatility*.

Definition 1. *Incremental mean volatility* of a real valued signal $s(t)$ is given by

$$\rho(s) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \|s(t+1) - s(t)\|. \tag{5}$$

III. Two Benchmark Real-Time Pricing Schemes

In this section, we discuss stability and price volatility issues of the electricity market. To that end, we present two real-time pricing and scheduling schemes that motivate the development in this paper. Both schemes are quite intuitive and even have been suggested as basic principles for real-time market operation. However, as we will demonstrate shortly, under flexible demand they lead to high volatility of price and supply, and then result in high supply cost.

In both schemes, we assume that the end users employ a threshold policy to determine whether they consume electricity at a particular time slot t. Specifically, consumer n compares the current price $p(t)$ to a scaled version of the backlog $\frac{q_n(t)}{\kappa_n^u}$, where κ_n^u is a constant. He applies a load of $x_n(t) = x_{n,max}$ if $p(t) \leq \frac{q_n(t)}{\kappa_n^u}$; otherwise he applies zero load, i.e. $x_n(t) = 0$. This threshold policy and similar threshold-based policies have been shown to be asymptotically optimal when the prices are exogenous [1], [6], [7]. In particular, if we consider the following optimization problem faced by a consumer

$$\min_{\{x_n(t)\}} \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[p(t) x_n(t)\right] \tag{6}$$

$$\text{s.t.} \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[x_n(t)\right] \geq \lambda_n, \ \forall n, \tag{7}$$

where $p(t)$ is the presumed i.i.d. price, one can show that if the prices $p(t)$ are exogenous, then the described threshold policy will asymptotically achieve the optimal value of (6) as κ_u^n is large. Since in this paper we assume that the consumers are price taking, we will use this threshold policy in both of the schemes studied below.

On the other hand, the two schemes differ in the way that the ISO sets prices as will see shortly.

$i)$ *Scheme I:* The ISO sets the price for the next time slot by $p(t+1) = \dot{C}\left(\sum_n x_n(t)\right)$, which is known as the *marginal price*. Note that if the ISO uses $\sum_n x_n(t)$ as a prediction of the load on the next time slot, then this price will also maximize the profit of the suppliers when the demand is inflexible, thus such a price-setting strategy appears to be quite reasonable. However, as we will demonstrate below, when the demand is flexible, high-volatile behavior will arise. We summarize the control decisions of the market participants in Scheme 1:

Scheme 1. *At time t:*

- *Consumer n computes:*

$$x_n(t) = x_{n,max} \mathbb{1}\left\{p(t) \leq \frac{q_n(t)}{\kappa_n^u}\right\} \tag{8}$$

$$q_n(t+1) = \left[q_n(t) + a_n(t) - x_n(t)\right]^+ \tag{9}$$

- *The ISO computes:*

$$p(t+1) = \dot{C}\left(\sum_n x_n(t)\right) \ \textit{and} \ s(t) = \sum_n x_n(t) \tag{10}$$

We show in Figure 2 the resulting evolution of price and total load under Scheme 1. As we can see from Figure 2, the resulting system dynamics under Scheme 1 are tremendously undesirable. Based on the threshold rule (8), the consumers use the maximum amount $x_{n,max}$ if the price is low and consume nothing if the price is high. Consequently, the aggregate load becomes either very large or too small.

We note that such opportunistically-aggressive behavior is commonly expected from cost-minimizing users in general. On the other hand, the ISO sets the price for the next slot to the marginal price which implicitly assumes that the load is static. Therefore, the price for the next slot is either high (if the total load is high) or low (if the total load is small). As a result, this mechanism creates violent fluctuations in both price and total load as depicted in Figure 2, and the system can not achieve a stable operating state.

Obviously, such a high-volatile load pattern can not minimize the generation cost in (6). Furthermore, the payments of users can be extremely low because the ISO basically follows the load one time slot behind in its price-setting rule. Thus, the payment from the consumers will not be able to offset the high generation cost either. Such detrimental behavior is the result of the aggregate opportunistic behavior of users with deferrable demand against a single market price that is determined myopically by the ISO ignoring the opportunistic consumer behavior.

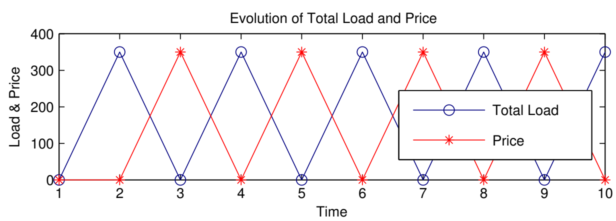

Fig. 2. Scheme 1 with 10 consumers having Poisson arrivals of demand with different rates, $\kappa_n^u = 100$, $C(x) = \frac{1}{2}x^2$. The Average supply cost is 30624 (optimum is 15312), price and total load volatility are both 350.

ii) Scheme II: The previous scheme reveals the undesirable market dynamics if the ISO directly uses the marginal price of supplying. One plausible direction to overcome this problem is to smoothen the price. For instance, we can calculate the price based on an iterative solution that solves an optimization problem. Specifically, note that from the convexity of the cost function, we can easily see that a lower bound for the minimum cost in (1) is given by a the solution to the following optimization problem.

$$\min_{\mathbf{x}} \quad C(s) \tag{11}$$

$$\text{s.t.} \quad \sum_{n=1}^{N} x_n \leq s \tag{12}$$

$$\lambda_n \leq x_n, \ \forall n \tag{13}$$

However, the values of λ_n are not known in advance, hence we can consider an iterative solution by means of a primal-dual approach, and obtain the following real-time pricing mechanism. Scheme 2, which is presented next, achieves the optimal solution of problem (11) asymptotically as $\kappa^s \to \infty$.

Scheme 2. *At time t:*

- *Consumer n computes:*

$$x_n(t) = x_{n,max} \mathbb{1}\left\{p(t) \leq \frac{q_n(t)}{\kappa_n^u}\right\} \tag{14}$$

$$q_n(t+1) = [q_n(t) + a_n(t) - x_n(t)]^+ \tag{15}$$

- *The supplier must meet the real load $\sum_n x_n(t)$. In addition it computes:*

$$s(t) = \dot{C}^{-1}(p(t)) \tag{16}$$

- *The ISO computes:*

$$p(t+1) = \left[p(t) + \kappa^s\left(\sum_n x_n(t) - s(t)\right)\right]^+ \tag{17}$$

Note that in (16), the suppliers compute a fictitious amount of supply $s(t)$ based on the current price although the real supply should always be equal to the instantaneous load. On the other hand, this fictitious supply $s(t)$ is used by the ISO to gradually adjust the price in (17). It may be possible to derive another algorithm that uses the real supply into (17) to update the price. However, the analysis will become more complicated, and the revised scheme will likely still lead to the volatility problem demonstrated below.

In this scheme, $s(t)$ changes slowly due to the dampening effect of κ^s, and price volatility is reduced compared to Scheme 1. However, supply cost is still high because of the abrupt fluctuations exhibited in load (cf. Figure 3). The reason behind the fluctuations in load is that the consumers still exhibit their opportunistic behavior despite the smaller fluctuations in price; the threshold rule (14) allows them to consume the maximum amount $x_{n,max}$ that they can when the price is at its lowest level.

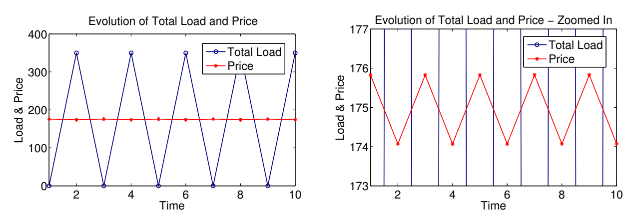

Fig. 3. Scheme 2 with 10 consumers having Poisson arrivals with different rates, $\kappa_n^u = 100$, $\kappa^s = 100$, $C(x) = \frac{1}{2}x^2$. Average supply cost is 30601 (optimum is 15312) and price volatility is 1.758.

IV. Distributed Real-Time Pricing and Scheduling Algorithm

Investigations in the previous section reveal that opportunistic consumer behavior resulting from demand flexibilities causes price and load volatility, which increases supply cost under real-time pricing mechanisms. In this section, we aim to design a new pricing scheme that not only is aligned with the customer's interest but also helps the ISO to minimize supply cost and mitigate the volatility in price and total load. To that end, we start with the static optimization in (11). Recall that problem (11) provides a lower bound to problem (1). We then augment the objective of problem (11) with an additional cost term motivated by the proximal optimization algorithm [8], [9]. Therefore, the optimization problem is

$$\min_{\mathbf{x},\mathbf{y}} \quad C(s) + \frac{\gamma}{2}\sum_{n=1}^{N}(x_n - y_n)^2 \tag{18}$$

$$\text{s.t.} \quad \sum_{n=1}^{N} x_n \le s \tag{19}$$

$$\lambda_n \le x_n, \ \forall n \tag{20}$$

where γ is a positive constant and $y_n \in \mathbb{R}$ are auxiliary variables. It is easy to see that if x_n^* and s^* are the optimal solution to problem (11), then $x_n = x_n^*$, $y_n = x_n^*$, and $s = s^*$ are trivially the optimal solution to problem (18). However, the quadratic term in (18) makes the problem strictly convex in x_n, which helps to alleviate volatile behaviors as we will see shortly. As in other proximal optimization algorithms [8], in each iteration we first fix $y_n(t)$ and optimize the objective of (18) over x_n. Let the corresponding optimal solution be $x_n(t)$. We then set $y_n(t+1) = x_n(t)$ and continue with the next iteration. By setting $y_n(t+1) = x_n(t)$, the quadratic term in (18) becomes $\frac{\gamma}{2}\sum_{n=1}^{N}\left(x_n(t) - x_n(t-1)\right)^2$, which penalizes the difference between the load at slot t and slot $t-1$. Accordingly, we propose the following new pricing mechanism for consumption:

New Consumer Pricing Mechanism: At each time t a price $p(t)$ is announced for each consumer n. The resulting payment at time t of the consumer to with a load $x_n(t)$ is:

$$p(t)x_n(t) + \frac{\gamma}{2}(x_n(t) - x_n(t-1))^2.$$

Here, the second term is the new component that incurs a penalty (with a fixed and uniform scaling γ across users) on the *change of load* for each consumer. Intuitively, this penalty encourages the flexible users to smooth out their loads and reduces the potential volatility. We note that this measure of change with respect to the last load can also be made over an average of the load levels of the consumer over a finite time horizon to have a similar effect. We will demonstrate soon that this simple pricing scheme is able to align users' interest with global grid stability.

For a fixed $y_n = x_n(t-1)$ we can write the Lagrangian function for the problem (18) w.r.t. x_n as:

$$L(\mathbf{x}, s, p, \mathbf{q}) = C(s) + \frac{\gamma}{2}\sum_{n=1}^{N}(x_n - y_n)^2 + p\left(\sum_{n=1}^{N} x_n - s\right) + \sum_{n=1}^{N} q_n(\lambda_n - x_n) \tag{21}$$

where $p \ge 0$ and $\mathbf{q} \triangleq [q_n \ge 0, n = 1, 2, \ldots, N]$ are the dual variables corresponding to the constraints in (19) and (20), respectively. Then, the dual function is given by

$$D(p, \mathbf{q}) = \sum_{n=1}^{N} \min_{x_n} \left\{ \left((p - q_n)x_n + \frac{\gamma}{2}(x_n - y_n)^2 \right) \right\} + \min_{s \ge 0} \{C(s) - ps\} + \sum_{n=1}^{N} \lambda_n q_n \tag{22}$$

The first optimization in (22) leads to the following intuitive interpretation of a new pricing scheme. We can interpret p as the price of electrical energy. Then, in addition to this common price, each user also receives a penalty for the deviation from its load in the previous time slot. The corresponding objective is strictly convex in x_n. Hence, if y_n is kept fixed, the solution to the sub-minimization is given by $x_n^* = \left[y_n + \frac{1}{\gamma}(q_n - p)\right]_0^{x_{max}}$, where the operator $[.]_a^b$ projects its argument onto the interval $[a, b]$. The second optimization in (22) is the profit maximization for the supplier. Then, the dual problem is simply $\max_{p \ge 0, \mathbf{q} \ge 0} \quad D(p, \mathbf{q})$.

Our distributed real-time pricing and scheduling algorithm then employs an iterative dual method where the primal variables $\mathbf{x}$, s and the dual variables p, $\mathbf{q}$ are updated at each iteration first with $\mathbf{y}$ kept fixed, and then $\mathbf{y}$ is updated at the end of each iteration by setting $\mathbf{y}(t+1) = \mathbf{x}(t)$. Hence, $y(t)$ is dropped from the algorithm since we replace it with $x(t-1)$. The algorithm is formally presented in the following.

Algorithm 1. *At iteration t:*

- *Consumer n takes a common price $p(t)$ plus a quadratic penalty parameter γ. Then, it computes its optimum load as*

$$x_n(t) = \left[x_n(t-1) + \frac{1}{\gamma}(q_n(t) - p(t))\right]_0^{x_{max}} \tag{23}$$

$$q_n(t+1) = [q_n(t) + \alpha(a_n(t) - x_n(t))]^+ \tag{24}$$

- *The supplier must meet the real load $\sum_n x_n(t)$. Further it computes:*

$$s(t) = \dot{C}^{-1}(p(t)) \tag{25}$$

- *The ISO computes:*

$$p(t+1) = \left[p(t) + \beta\left(\sum_{n=1}^{N} x_n(t) - s(t)\right)\right]^+ \tag{26}$$

We note that Algorithm 1 is similar to the proposed algorithm in [9] in the sense that it takes one step towards the solution of the optimization (18)-(20) with $\mathbf{y}$ kept fixed. One difference, though, is that our algorithm directly sets $y_n(t) = x_n(t-1)$, bypassing another optimization step involving the recently updated dual variables [9]. However, this small difference changes the behavior of the algorithm significantly from that of [9]. In particular, while the algorithm in [9] converges for each variable, our algorithm does not converge for each x_n. Indeed, we will show later that each consumer's load $x_n(t)$ still exhibits some sinusoid-like behavior. On the other hand, our proposed pricing scheme with quadratic penalties ensures that the overall load is stable, and thus prevent the system instability reported in existing schemes in Section III.

In the following subsection, we propose a continous-time approximation of the dynamics of Algorithm 1 and investigate its convergence properties, with an emphasis on the convergence of the total load, total generation, and price.

A. A Continuous-Time Approximation and its Convergence

In order to study the dynamics and convergence of Algorithm 1, we first present a continuous-time approximation of the algorithm, investigate its convergence, and relate our findings to the original discrete-time algorithm. In Algorithm 1, as the step sizes α and β become very small the difference equations can be approximated by differential equations given below.

Algorithm 2 (**Algorithm A1-C**). *The continuous-time algorithm is governed by the following differential equations:*

$$\dot{x}_n(t) = \begin{cases} \frac{1}{\gamma}(q_n(t) - p(t)) & \textit{if } {}^{x_n(t)>0, \textit{ or}}_{q_n(t)-p(t)\geq 0} \\ 0 & \textit{otherwise} \end{cases} \tag{27}$$

$$\dot{q}_n(t) = \begin{cases} \alpha(\lambda_n - x_n(t)) & \textit{if } {}^{q_n(t)>0, \textit{ or}}_{\lambda_n - x_n(t)\geq 0} \\ 0 & \textit{otherwise} \end{cases} \tag{28}$$

$$s(t) = \dot{C}^{-1}(p(t)) \tag{29}$$

$$\dot{p}(t) = \begin{cases} \beta \left(\sum_{n=1}^{N} x_n(t) - s(t) \right) & \textit{if } {}^{p(t)>0, \textit{ or}}_{\left(\sum_{n=1}^{N} x_n(t) - s(t)\right)>0} \\ 0 & \textit{otherwise} \end{cases} \tag{30}$$

Before beginning the convergence study of Algorithm 2 dynamics, we give a definition of a stationary point of the sum of variables.

Definition 2. *Define* $\Phi(t) \triangleq (\sum_n x_n(t), \sum_n q_n(t), p(t), s(t))$. $\Phi^* \triangleq (X^*, Q^*, p^*, s^*)$ *is a stationary point of Algorithm 2 in the sum sense, if* $\Phi(t_0) = \Phi^*$ *for some* $t_0 < \infty$ *and* $\Phi(t) = \Phi(t_0)$ *for all* $t > t_0$.

The following proposition states that the system of equations given in **A1-C** has equilibrium properties for the sums of user loads $\sum_n x_n(t)$, the sums of user queues $\sum_n q_n(t)$, the price $p(t)$, and the total supply $s(t)$.

Proposition 1. *In the system characterized by the continuous-time algorithm **A1-C**,* $\Phi(t)$ *converges to a stationary point* Φ^*. *Furthermore,* Φ^* *is a stationary point of the discrete-time Algorithm 1.*

Proof. To begin with, we define the system state at time t as $\Theta(t) \triangleq (\vec{x}(t), \vec{q}(t), p(t), s(t))$. We consider the following Lyapunov function:

$$V(\Theta(t)) = \frac{N}{2\alpha}(p(t) - p^*)^2 + \frac{\gamma}{2}\left(\sum_n x_n(t) - X^*\right)^2 + \frac{1}{2\beta}\left(\sum_n q_n(t) - Q^*\right)^2 \tag{31}$$

where $\Phi^* \triangleq (X^*, Q^*, p^*, s^*)$ is a stationary point of Algorithm 1 as described in Definition 2. We did not include $s(t)$ in the Lyapunov function, because it is solely determined by $p(t)$. The derivative of $V(\Theta(t))$ w.r.t. time t is

$$\dot{V}(\Theta(t)) = \gamma\left(\sum_n x_n(t) - X^*\right)\sum_n \dot{x}_n(t) + \frac{1}{\beta}\left(\sum_n q_n(t) - Q^*\right)\sum_n \dot{q}_n(t) + \frac{N}{\alpha}(p(t) - p^*)\,\dot{p}(t) \tag{32}$$

$$\leq \left(\sum_n x_n(t) - X^*\right)\sum_n (q_n(t) - p(t)) + \left(\sum_n q_n(t) - Q^*\right)\sum_n (\lambda_n - x_n(t)) + N(p(t) - p^*)\left(\sum_n x_n(t) - s(t)\right) \tag{33}$$

where the derivatives are replaced with their corresponding expressions in (27)-(30). Adding and subtracting the stationary values of Φ^* in (33), we obtain

$$\begin{aligned}
&\dot{V}(\Theta(t)) \\
&\leq \left(\sum_n x_n(t) - X^*\right)\left(\sum_n q_n(t) - Q^* + Np^* - Np(t)\right) \\
&\quad + \left(\sum_n q_n(t) - Q^*\right)\left(\sum_n \lambda_n - X^* + X^* - \sum_n x_n(t)\right) \\
&\quad + (p(t) - p^*)\left(\sum_n x_n(t) - X^* + s^* - s(t)\right) \qquad (34)\\
&= \left(\sum_n x_n(t) - X^*\right)\left(\sum_n q_n(t) - Q^*\right) \\
&\quad + \left(\sum_n x_n(t) - X^*\right) N(p^* - p(t)) \\
&\quad + \left(\sum_n q_n(t) - Q^*\right)\left(X^* - \sum_n x_n(t)\right) \\
&\quad + N(p(t) - p^*)\left(\left(\sum_n x_n(t) - X^*\right) + (s^* - s(t))\right) \qquad (35)\\
&= N(p(t) - p^*)(s^* - s(t)) \qquad (36)
\end{aligned}$$

In (34), we used $Q^* = Np^*$, $s^* = X^* = \sum_n \lambda_n$, which follows from Definition 2, and then we obtained (36) by canceling the terms in (35). Noting that $p(t) = \dot{C}(s(t))$ and C is convex, we conclude

$$\dot{V}(\Theta(t)) \leq N(p(t) - p^*)(s^* - s(t)) \leq 0. \tag{37}$$

Now, we will show that $\lim_{t\to\infty} V(\Theta(t)) = 0$ and that the algorithm converges to Φ^*. First, we define an *invariant* set with respect to algorithm **A1-C** to be the set of states such that any trajectory with an initial point in this set indefinitely remains in it. We also define $\mathcal{S} \triangleq \left\{\Theta(t) : \dot{V}(\Theta(t)) = 0\right\}$ to be the set of states with the derivative of the Lyapunov function $V(\Theta(t))$ is 0, and $\mathcal{I}$ to be the largest invariant set in $\mathcal{S}$. Since $\dot{V}(\Theta(t)) \leq 0$ whenever Θ is outside of $\mathcal{S}$, by LaSalle's principle [10] any trajectory $\{\Theta(t), t \geq 0\}$ of the algorithm asymptotically approaches to the set $\mathcal{I}$.

It remains to show that the invariant set $\mathcal{I}$ consists of the points which are stationary points of Algorithm 1 (c.f. Definition 2). Formally, we want to show that

$$\mathcal{I} = \left\{\Theta \in \mathcal{S} : \left(\sum_n x_n(t), \sum_n q_n(t), p(t), s(t)\right) = \Phi^*\right\} \tag{38}$$

Let $\Theta(0) \in \mathcal{I}$. Since $\dot{V}(\Theta(t)) = 0$ in $\mathcal{I}$, $p(t) = p^*$ and $s(t) = s^*$ for $t \geq 0$ due to (37). Hence, $\dot{p}(t) = 0$ and $\sum_n x_n(t) = s^* = X^*$ for $t \geq 0$ from (30). Observing (31), and noting $\sum_n \dot{x}_n(t) = 0$ and $\dot{p}(t) = 0$, we have $\sum_n \dot{q}_n(t) = 0$. Thus, it remains to show that $\sum_n q_n(t) = Q^*$.

Let $\mathcal{N}_1(t) \triangleq \{n : \dot{x}_n(t) \neq 0\}$ and $\mathcal{N}_2(t) \triangleq \{n : \dot{x}_n(t) = 0\}$. We assume that $|\mathcal{N}_2(t)| > 0$, and we will show a contradiction in the following. First, we note that $x_n(t) = 0$ for $n \in \mathcal{N}_2(t)$, and then plug it in (28) to get $\sum_{\mathcal{N}_2(t)} \dot{q}_n(t) = \alpha \sum_{\mathcal{N}_2(t)} \lambda_n > 0$. For sufficiently small α we can assure, for $n \in \mathcal{N}_2(t)$, that $q_n(t + \delta t) < p^*$ and $x_n(t + \delta t) = 0$, and consequently that $n \in \mathcal{N}_2(t + \delta t)$.

For $n \in \mathcal{N}_1(t)$ consider two cases: (i) $\mathcal{N}_1(t) = \mathcal{N}_1(t+\delta t)$, and (ii) $\mathcal{N}_1(t + \delta t) \subset \mathcal{N}_1(t)$, i.e. some $n \in \mathcal{N}_1(t)$ moves to $\mathcal{N}_2(t + \delta t)$.

For case (i), summing (27) over all n we obtain $\sum_{\mathcal{N}_1(t)} q_n(t) = \sum_{\mathcal{N}_1(t+\delta t)} q_n(t + \delta t) = |\mathcal{N}_1(t)|p^*$, and hence $\sum_{\mathcal{N}_1(t)} \dot{q}_n(t) = 0$, which is contradictory because $\sum_n \dot{q}_n(t) = \sum_{\mathcal{N}_1(t)} \dot{q}_n(t) + \sum_{\mathcal{N}_2(t)} \dot{q}_n(t) = -\alpha \sum_{\mathcal{N}_2(t)} \lambda_n \neq 0$.

For case (ii), let $\mathcal{M} \triangleq \mathcal{N}_1(t) \setminus \mathcal{N}_1(t + \delta t)$. Then we have

$$\begin{aligned}
&\sum_n q_n(t + \delta t) - \sum_n q_n(t) \\
&= \sum_{\mathcal{N}_1(t+\delta t)} q_n(t + \delta t) - \sum_{\mathcal{N}_1(t)} q_n(t) \\
&\quad + \sum_{\mathcal{N}_2(t+\delta t)} q_n(t + \delta t) - \sum_{\mathcal{N}_2(t)} q_n(t) &(39)\\
&= |\mathcal{N}_1(t + \delta t)|p^* - |\mathcal{N}_1(t)|p^* \\
&\quad + \sum_{\mathcal{N}_2(t)} (q_n(t + \delta t) - q_n(t)) + \sum_{\mathcal{M}} q_n(t + \delta t) &(40)\\
&= -|\mathcal{M}|p^* + \sum_{\mathcal{N}_2(t)} (q_n(t + \delta t) - q_n(t)) \\
&\quad + \sum_{\mathcal{M}} (q_n(t + \delta t) - q_n(t)) + \sum_{\mathcal{M}} q_n(t) &(41)
\end{aligned}$$

where we used $\sum_{\mathcal{N}_1(t+\delta t)} q_n(t+\delta t) = |\mathcal{N}_1(t+\delta t)|p^* > 0$ to obtain (40). Dividing by δt, and taking the limit as $\delta t \to 0$, we get

$$\sum_n \dot{q}_n(t) = \sum_{\mathcal{N}_2(t)} \dot{q}_n(t) + \sum_{\mathcal{M}} \dot{q}_n(t) \qquad (42)$$

$$0 = \sum_{\mathcal{N}_1(t+\delta t)} \dot{q}_n(t) \qquad (43)$$

where (43) is obtained by subtracting $\sum_n \dot{q}_n(t)$ from both sides of (42).

From (27), $q_n(t) \geq p^*$ implies $\dot{x}_n(t) \geq 0$. However, if $x_n(t) > 0$ we have $n \in \mathcal{N}_1(t + \delta t)$; if $x_n(t) = 0$ we have $\dot{q}_n(t) > 0$ and thus $n \in \mathcal{N}_1(t + \delta t)$. Therefore $q_n(t) < p^*$ for $n \in \mathcal{M}$, and consequently $\sum_{\mathcal{N}_1(t)\setminus\mathcal{M}} q_n(t) = \sum_{\mathcal{N}_1(t+\delta t)} q_n(t) > |\mathcal{N}_1(t) \setminus \mathcal{M}|p^* = |\mathcal{N}_1(t + \delta t)|p^*$. However, $\sum_{\mathcal{N}_1(t+\delta t)} q_n(t + \delta t) = |\mathcal{N}_1(t + \delta t)|p^*$ implying $\sum_{\mathcal{N}_1(t+\delta t)} \dot{q}_n(t) < 0$, which is in contradiction with (42).

As a result $\mathcal{N}_2 = \emptyset$, and $|\mathcal{N}_1| = N$. Therefore, $\sum_n q_n(t) = \sum_{\mathcal{N}_1} q_n(t) = Np^* = Q^*$. □

It can be deduced from Proposition 1 that the volatility of price and total load converge to zero under the real-time pricing algorithm. However, our algorithm does not converge for each consumer's load x_n. The following corollary states that each $x_n(t)$ exhibits a sinusoid-like behavior at its stationary operating regime (X^*, Q^*, p^*, s^*).

Corollary 1. *If the initial state of algorithm* ***A1-C*** *is* $(\vec{x}(0), \vec{q}(0), s(0), p(0)) = (\vec{x}, \vec{q}, s^*, p^*)$ *such that* $\sum_n x_n = X^*$ *and* $\sum_n q_n = Q^*$*, then the trajectories of* $q_n(t)$ *and* $x_n(t)$ *are characterized by*

$$\begin{aligned}
q_n(t) &= p^* + (q_n(0) - p^*)\cos(\sqrt{\alpha/\gamma}t) \\
&\quad - \sqrt{\alpha\gamma}(x_n(0) - \lambda_n)\sin(\sqrt{\alpha/\gamma}t) &(44)\\
x_n(t) &= \lambda_n + (x_n(0) - \lambda_n)\cos(\sqrt{\alpha/\gamma}t) \\
&\quad + \frac{1}{\sqrt{\alpha\gamma}}(q_n(0) - p^*)\sin(\sqrt{\alpha/\gamma}t). &(45)
\end{aligned}$$

In Figure 4, the sinusoid-like behavior described in Corollary 1 is depicted. Although the load amount $x_n(t)$ of each consumer depends on his own queue length $q_n(t)$, they evolve asynchronously in time as seen in Figure 4 such that the total load is constant.

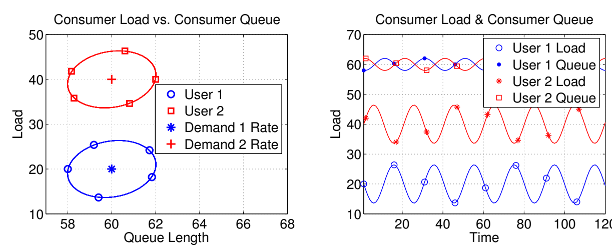

Fig. 4. Equilibrium behavior of Algorithm 1 with 2 consumers having constant demand arrivals, $\gamma = 1$, $\alpha = 0.1$, $\beta = 0.01$, $C(x) = \frac{1}{2}x^2$.

V. Numerical Results

In this section, we first demonstrate the convergence of the real-time pricing algorithm, and its evolution in the stationary regime under stochastic demand arrivals. Then, we compare the cost, volatility, and ISO deficit (to be defined soon) of the algorithm to those of Schemes 1 & 2.

The behavior of the system governed by Algorithm 1 under Poisson distributed demand arrivals is depicted in Figure 5. Observe that the price and the total load vary in time inside a relatively small margin around the stationary values p^* and X^*, respectively. We note that the convergence of iterative algorithms under stochastic dynamics usually requires decreasing step sizes. Nevertheless, individual consumer loads $x_n(t)$ exhibit a sinusoid-like behavior in an asynchronous manner so that the total does not change considerably.

In Figure 6, the price and total load volatility achieved by Algorithm 1 and their effect on the supply cost are compared to those of Schemes 1 & 2. As we already have seen in Figure 2-3, the total load volatility is quite large for both schemes, and only the price volatility is reduced in Scheme 2. Along with these observations, Figure 6 reveals that both price and total load volatility are significantly reduced by

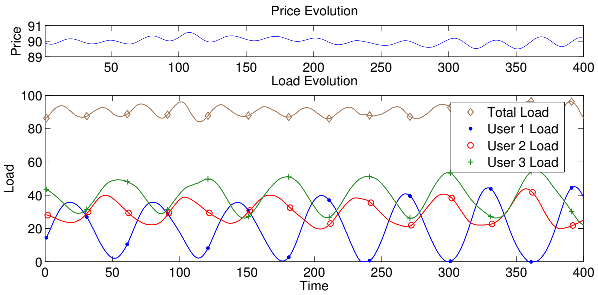

Fig. 5. System evolution under Algorithm 1 with Poisson distributed demand arrivlas, $\gamma = 1$, $\alpha = 0.01$, $\beta = 0.01$, $C(x) = \frac{1}{2}x^2$.

Algorithm 1 even under stochastic demand. The second plot in Figure 6 demonstrates the effect of the total load volatility on supply cost. For a convex cost function, volatility serves as an indicator of the supply cost. We observe in Figure 6 that supply cost is reduced under Algorithm 1 together with the total load volatility, whereas supply cost is high due to high load volatility in Schemes 1 & 2.

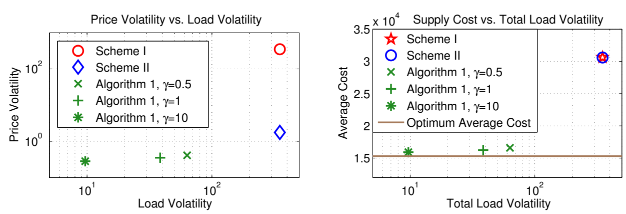

Fig. 6. Comparison of Algorithm 1 to Scheme 1-2 in terms of average supply cost and volatility under Poisson distributed arrivals. For Scheme 1-2 $\kappa_n^u = 100$, $\kappa^s = 100$, for Algorithm 1 $\alpha = 1$, $\beta = 0.01$, and $C(x) = \frac{1}{2}x^2$

Next, we numerically investigate the ISO deficit, which is defined as the difference between the payments from consumers and the payments to suppliers. Remember that in both Schemes 1-& 2 and Algorithm 1 the suppliers must always follow the load generated by the consumers, even though the current price does not maximize their profit. However, the *actual* payments made by the ISO to the suppliers should also coincide with suppliers' interests. Therefore, we assume that the ISO pays the suppliers the marginal cost of supplying the realized demand. The resulting difference between the payments from consumers and the actual payment made to the suppliers is depicted in Figure 7.

We observe in Figure 7 that the payments to suppliers and the payment from consumers are quite close to each other under Algorithm 1, whereas in Schemes 1 & 2, they are far from balancing each other due to high volatility of price and total load. In the second plot in Figure 7, we plot the ISO deficit for various values of the temporal penalty parameter γ. Figure 7 suggests that γ can be adjusted so that the ISO deficit can be reduced to zero. This observation raises the interesting question whether iteratively updating γ can help reduce the ISO deficit, which will be addressed in our future work.

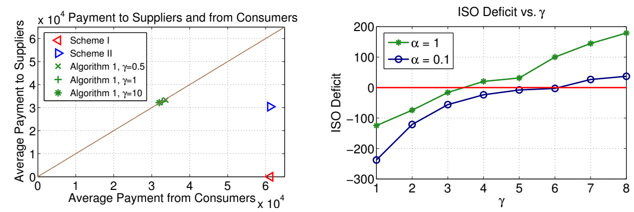

Fig. 7. Comparison of Algorithm 1 to Scheme 1-2 in terms of the ISO deficit under Poisson distributed arrivals. For Scheme 1-2 $\kappa_n^u = 100$, $\kappa^s = 100$, for Algorithm 1 $\alpha = 1$, $\beta = 0.01$, and $C(x) = \frac{1}{2}x^2$

VI. Conclusions

We have investigated the design of pricing mechanisms for the stable and cost-effective service of opportunistic consumers with deferrable demands in the electricity market. Our initial investigations have revealed the significant instability issues that arise from improper handling of economically-driven consumers' opportnistic behavior. We have proposed a novel pricing basd on a proximal optimization formulation that not only prices the absolute consumption but also *the change in the consumption* of users, which resolves the volatility and market clearance issues of the benchmark strategies. Our proposed pricing solution is: simple in that consumers need not predict ex-post price corrections or other users' behavior (which is highly complex to estimate in a large heterogeneous demand market); and fair in that all consumers are subject to the same pricing levels at all times; and market clearing in that by adjusting the change of load penalty level, the supply and sonsumption payments can be balanced. The main insights derived from this study is expected to be more generally applicable to a wider class of demand flexibilities, and suggests a promising means of harnessing the potential gains of demand flexibilities without suffering from the detrimental impact of opportunistic consumer behavior.

References

[1] M. J. Neely, A. S. Tehrani, and A. G. Dimakis, "Efficient algorithms for renewable energy allocation to delay tolerant consumers," in *Smart Grid Communications (SmartGridComm), 2010 First IEEE International Conference on*. IEEE, 2010, pp. 549–554.

[2] D. Materassi, M. Roozbehani, and M. A. Dahleh, "Equilibrium price distributions in energy markets with shiftable demand," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*. IEEE, 2012, pp. 3183–3188.

[3] M. Roozbehani, M. A. Dahleh, and S. K. Mitter, "Volatility of power grids under real-time pricing," *Power Systems, IEEE Transactions on*, vol. 27, no. 4, pp. 1926–1940, 2012.

[4] M. Roozbehani, M. Rinehart, M. Dahleh, S. Mitter, D. Obradovic, and H. Mangesius, "Analysis of competitive electricity markets under a new model of real-time retail pricing," in *Energy Market (EEM), 2011 8th International Conference on the European*. IEEE, 2011, pp. 250–255.

[5] A.-H. Mohsenian-Rad and A. Leon-Garcia, "Optimal residential load control with price prediction in real-time electricity pricing environments," *Smart Grid, IEEE Transactions on*, vol. 1, no. 2, pp. 120–133, 2010.

[6] S. Chen, P. Sinha, and N. B. Shroff, "Scheduling heterogeneous delay tolerant tasks in smart grid with renewable energy," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*. IEEE, 2012, pp. 1130–1135.

[7] T. T. Kim and H. V. Poor, "Scheduling power consumption with price uncertainty," *Smart Grid, IEEE Transactions on*, vol. 2, no. 3, pp. 519–527, 2011.
[8] D. P. Bertsekas and J. N. Tsitsiklis, "Parallel and distributed computation," 1989.
[9] X. Lin and N. B. Shroff, "Utility maximization for communication networks with multipath routing," *Automatic Control, IEEE Transactions on*, vol. 51, no. 5, pp. 766–781, 2006.
[10] H. K. Khalil, *Nonlinear systems*. Prentice hall Upper Saddle River, 2002, vol. 3.