Solution to HW5

Monday, April 10, 2023 3:49 PM

① The state: $(h_1(t), h_2(t))$, which are the current AoI of the two sensors

The action $u=1$, if sensor 1 is scheduled for transmission; of $u=2$, if sensor 2 is scheduled for transmission.

The transition probability:

- If $u=1$,

$$(h_1(t+1), h_2(t+1)) = \begin{cases} (1, h_2(t)+1) & \text{with prob. } p_1 \\ (h_1(t)+1, h_2(t)+1) & \text{prob. } 1-p_1 \end{cases}$$

- If $u=2$,

$$(h_1(t+1), h_2(t+1)) = \begin{cases} (h_1(t)+1, 1) & \text{with prob. } p_2 \\ (h_1(t)+1, h_2(t)+1) & \text{w. prob. } (1-p_2) \end{cases}$$

Cost: $h_1(t) + h_2(t)$

② The Bellman Equation:

$$J(h_1, h_2) = \min \begin{cases} h_1 + h_2 + \gamma \left[p_1 J(1, h_2+1) + (1-p_1) J(h_1+1, h_2+1) \right] \\ h_1 + h_2 + \gamma \left[p_2 J(h_1+1, 1) + (1-p_2) J(h_1+1, h_2+1) \right] \end{cases}.$$

③ The pseudo-code

Max_State = 100

J(1: Max_State, 1: Max_State, 1: Num_Iterations) = 0

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for i = 1 to Num_Iterations - 1
    for h1 = 1 to Max_State
        for h2 = 1 to Max_State
            next_h1 = min(h1+1, Max_State)
            next_h2 = min(h2+1, Max_State)
            Cost_to_go1 = h1 + h2 + γ[p1 J(1, next_h2)
                          + (1-p1) J(next_h1, next_h2)]
            Cost_to_go2 = h1 + h2 + γ[p2 J(next_h1, 1)
                          + (1-p2) J(next_h1, next_h2)]
            J(h1, h2, i+1) = min(Cost_to_go1,
                                 Cost_to_go2)
        end
    end
end
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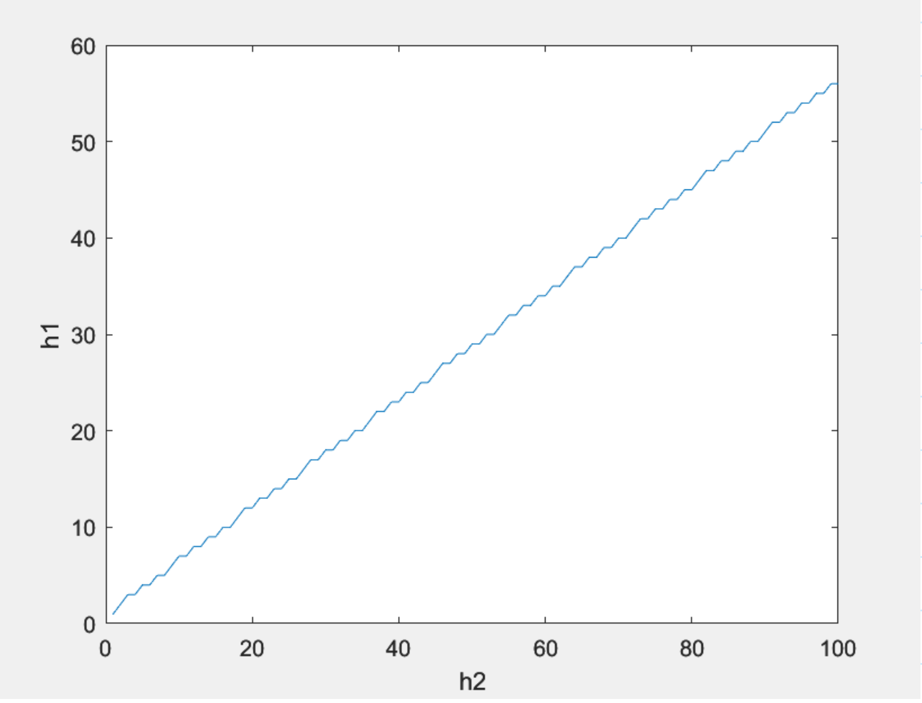
(4) The following figure shows the threshold for h_1 as a function of h_2.



(5) First, we show that $J(h_1, h_2)$ is non-decreasing in either h_1 to h_2. To see this, let us compare $J(h_1, h_2)$ and $J(h_1+1, h_2)$. By definition

$$J(h_1,h_2) = \min_{\mu} E\left[\sum_{t=1}^{+\infty} \gamma^{t-1}(h_1(t)+h_2(t)) \middle| \begin{array}{l} h_1(1)=h_1 \\ h_2(1)=h_2 \end{array} \right]$$

$$J(h_1+1,h_2) = \min_{\mu} E\left[\sum_{t=1}^{+\infty} \gamma^{t-1}(h_1(t)+h_2(t)) \middle| \begin{array}{l} h_1(1)=h_1+1 \\ h_2(1)=h_2 \end{array} \right]$$

For the same sequence of scheduling decisions & the same realization of transmission successes, it is easy to verify that

$$h_1(t)\Big|_{h_1(1)=h_1+1} \geq h_1(t)\Big|_{h_1(t)=h_1}$$

$$h_2(t)\Big|_{h_1(1)=h_1+1} = h_2(t)\Big|_{h_1(t)=h_1}$$

Therefore, we must have $J(h_1+1, h_2) \geq J(h_1, h_2)$

Similarly, we can show that $J(h_1, h_2+1) \geq J(h_1, h_2)$

Next, suppose that for a given pair h_1, h_2, the optimal decision is to schedule sensor 1. This implies that

$$\text{Cost-to-go 1} \leq \text{Cost-to-go 2}$$

i.e.

$$h_1+h_2+\gamma\left[p_1 J(1, h_2+1) + (1-p_1) J(h_1+1, h_2+1)\right]$$

$$\leq h_1+h_2+\gamma\left[p_2 J(h_1+1, 1) + (1-p_2) J(h_1+1, h_2+1)\right] \quad (*)$$

Now, suppose that the current state is changed to h_1+1, h_2. We wish to compare

Now, suppose that the current state is changed to h_1+1, h_2. We wish to compare

$$\text{Cost\_to\_go1} - \text{Cost\_to\_go2}$$

$$= (h_1+1) + h_2 + \gamma\left[p_1 J(1, h_2+1) + (1-p_1) J(h_1+2, h_2+1)\right]$$

$$- \left\{(h_1+1) + h_2 + \gamma\left[p_2 J(h_1+2, 1) + (1-p_2) J(h_1+2, h_2+1)\right]\right.$$

$$\leq \gamma(1-p_1)\left[J(h_1+2, h_2+1) - J(h_1+1, h_2+1)\right]$$

$$- \gamma p_2\left[J(h_1+2, 1) - J(h_1+1, 1)\right]$$

$$- \gamma(1-p_2)\left[J(h_1+2, h_2+1) - J(h_1+1, h_2+1)\right]$$

(using (*))

$$= \gamma(\underbrace{p_2-p_1}_{\leq 0})\left[\underbrace{J(h_1+2, h_2+1) - J(h_1+1, h_2+1)}_{\geq 0}\right]$$

$$- \gamma \underbrace{p_2}_{\geq 0}\left[\underbrace{J(h_1+2, 1) - J(h_1+1, 1)}_{\geq 0}\right]$$

$$\leq 0$$

Therefore, the optimal decision for (h_1+1, h_2) is still scheduling sensor 1.

This shows that there must be a threshold H_1 such that sensor 1 is scheduled for any $h_1 \geq H_1$ and it is not scheduled for any $h_1 < H_1$. The threshold depends on h_2.

(6) The only change to the Bellman equation or the pseudo code is to change

(6) The only change to the Bellman equation or the pseudo code is to change

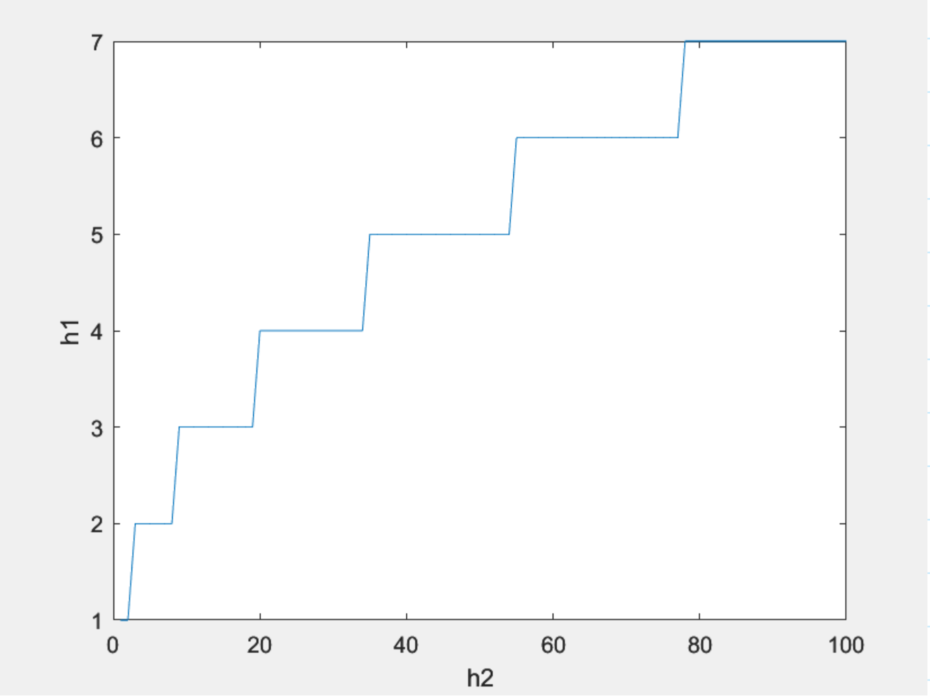
$$h_1 + h_2$$

to

$$h_1^2 + h_2 .$$

Since these terms are cancelled in (*), the threshold structure should still work.

This figure shows the threshold for h_1, as a function of h_2.



We can see that the threshold for scheduling sensor 1 is a lot lower, because we assign a higher cost for sensor 1's AoI.