Solution to homework #5

2. i) $g(\lambda) = \min_x L(x, \lambda)$

where $L(x,\lambda) = U(x) + \lambda(x-R)$ is a linear function of λ. Hence, $g(\lambda)$ is concave since it is the pointwise minimum of linear functions.

ii) We need to show that for all λ_1, λ_2,

$$[(-X_0(\lambda_1)+R) - (-X_0(\lambda_2)+R)](\lambda_1-\lambda_2)$$
$$\geq M\,\|(-X_0(\lambda_1)+R) - (-X_0(\lambda_2)+R)\|^2$$

$$\Leftrightarrow \quad -[X_0(\lambda_1) - X_0(\lambda_2)](\lambda_1-\lambda_2)$$
$$\geq M\,\|X_0(\lambda_1) - X_0(\lambda_2)\|^2$$

Now, since $X_0(\lambda)$ minimizes $U(x) + \lambda(x-R)$, we have,

$$U'(X_0(\lambda_1)) + \lambda_1 = 0$$
$$U'(X_0(\lambda_2)) + \lambda_2 = 0.$$

Since $U''(x) \geq M$, we have

$$-(\lambda_1-\lambda_2)\,(X_0(\lambda_1) - X_0(\lambda_2))$$
$$= [U'(X_0(\lambda_1)) - U'(X_0(\lambda_2))]\,(X_0(\lambda_1) - X_0(\lambda_2))$$
$$\geq M\,\|X_0(\lambda_1) - X_0(\lambda_2)\|^2$$

iii) To show Lipschitz condition, note that

$$M\,\|\partial g(\lambda_1) - \partial g(\lambda_2)\|^2 \leq (\partial g(\lambda_1) - \partial g(\lambda_2))^T(\lambda_1-\lambda_2)$$
$$\leq \|\partial g(\lambda_1) - \partial g(\lambda_2)\| \cdot \|\lambda_1 - \lambda_2\|$$
$$\Rightarrow \|\partial g(\lambda_1) - \partial g(\lambda_2)\| \leq \frac{1}{M}\|\lambda_1 - \lambda_2\|.$$

4. (a) $$\max \sum_{i=1}^{n} \alpha_i W_i \log\left(1+\frac{\beta_i P_i}{W_i}\right)$$

perspective of a concave function $\log(1+\beta_i P_i)$.

Sub to $$\sum_{i=1}^{n} P_i = P_0$$

$$\sum_{i=1}^{n} W_i = W_0$$

$$P_i \geq 0, \quad W_i \geq 0$$

This is a convex program because the objective function is concave, and the constraint set is convex.

(b)(c) Associate a Lagrange multiplier for the constraint $\sum_{i=1}^{n} P_i = P_0$. The Lagrangian is

$$L(\vec{P}, \vec{w}, \lambda) = -\sum_{i=1}^{n} \alpha_i W_i \log\left(1+\frac{\beta_i P_i}{W_i}\right) + \lambda\left(\sum_{i=1}^{n} P_i - P_0\right)$$

$$= -\sum_{i=1}^{n}\left[\alpha_i W_i \log\left(1+\frac{\beta_i P_i}{W_i}\right) - \lambda P_i\right] - \lambda P_0$$

Since the Slater condition holds, there must exist a Lagrange multiplier λ such that $\vec{P}, \vec{w}$ minimizes $L(\vec{P}, \vec{w}, \lambda)$, over all $P_i \geq 0$ and $\sum_{i=1}^{n} W_i = W_0$.

Since $$\frac{\partial L}{\partial P_i} = -\frac{\alpha_i \beta_i}{1+\frac{\beta_i P_i}{W_i}} + \lambda$$

Hence P_i must satisfy either

$$-\frac{\alpha_i \beta_i}{1+\frac{\beta_i P_i}{W_i}} + \lambda = 0, \quad \text{and } P_i \geq 0$$

or $P_i = 0$. We then have

$$P_i = \begin{cases} \frac{W_i}{\beta_i}\left(\frac{\alpha_i \beta_i}{\lambda} - 1\right) & \text{if } \frac{\alpha_i \beta_i}{\lambda} \geq 1 \\ 0 & \text{otherwise} \end{cases} \qquad (*)$$

Let $J = \{ i \mid \frac{\alpha_i \beta_i}{\lambda} - 1 \geq 0 \}$, then when (*) is true

$$L(\vec{p}, \vec{w}, \lambda) = \sum_{i \in J} \left[-\alpha_i w_i \log \frac{\alpha_i \beta_i}{\lambda} + \frac{\lambda w_i}{\beta_i} \left(\frac{\alpha_j \beta_i}{\lambda} - 1 \right) \right] - \lambda p_0$$

Since this is linear in $\vec{w}$, we have

$w_i > 0$ only if its coefficient

$\alpha_i \log \frac{\alpha_i \beta_i}{\lambda} - \frac{\lambda}{\beta_i} \left(\frac{\alpha_i \beta_i}{\lambda} - 1 \right)$ is the

largest among all $i \in J$.

Note: Here taking $\frac{\partial L}{\partial w_i} = 0$ does not help because equality only holds when $w_i > 0$.

Finally, if $\lambda \leq 0$, then by (*) we can't have

$$\sum_{i=1}^{n} p_i = p_0$$

Hence $\lambda > 0$.

(d) Using gradient-ascent on the dual, we can the obtain the following control algorithm.

At each iteration t:

(i) Among those receivers with $\frac{\alpha_i \beta_i}{\lambda(t)} > 1$, pick one receiver i with the largest value of

$$\alpha_i \log \frac{\alpha_i \beta_i}{\lambda(t)} - \frac{\lambda(t)}{\beta_i} \left(\frac{\alpha_i \beta_i}{\lambda(t)} - 1 \right).$$

Assign $w_i(t) = w_0$ for this receiver, and assign $w_j(t) = 0$ for all other receivers $j \neq i$. Let $w(t) = w_0$.

If there is no receiver with $\frac{\alpha_i \beta_i}{\lambda(t)} > 1$, let $w(t) = 0$.

(ii) For the receiver i chosen in step (i), assign power

$$p(t) = \frac{W(t)}{\beta_i}\left(\frac{\alpha_i \beta_i}{\lambda(t)} - 1\right).$$

(iii) Update the dual variable by

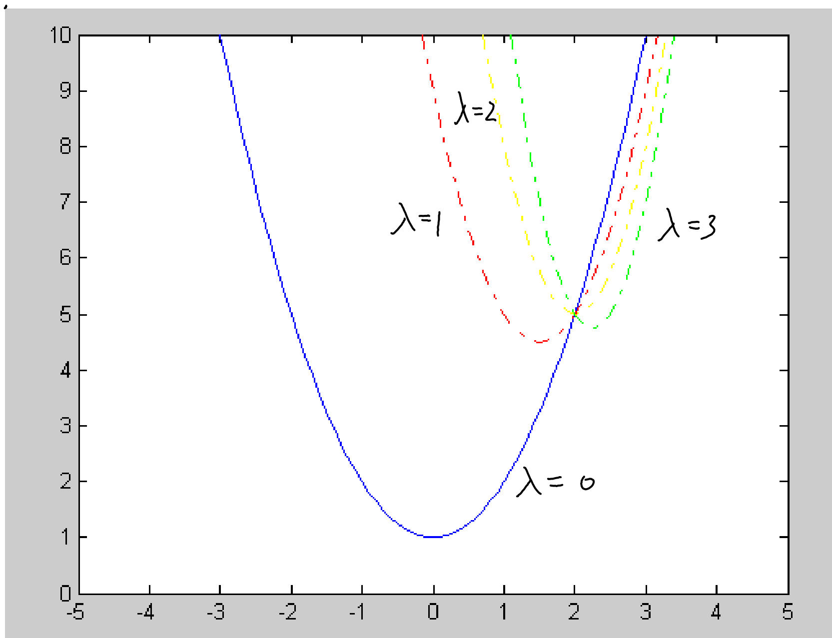
$$\lambda(t+1) = \left\{\lambda(t) + \gamma\left[p(t) - p_0\right]\right\}^+.$$

Problem 5.1

[redacted]

(a) The feasible set is $x \in [2, 4]$
The optimal solution is $x^* = 2$
The optimal value is $p^* = (x^*)^2 + 1 = 5$

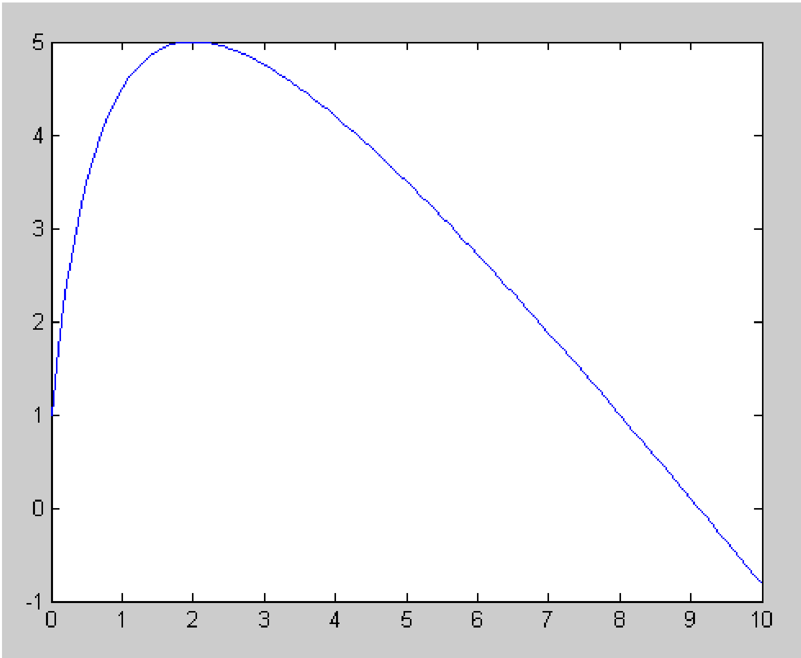
(b) $$L(x, \lambda) = x^2 + 1 + \lambda(x-2)(x-4)$$
$$= (1+\lambda)x^2 - 6\lambda x + (1 + 8\lambda)$$
$$= (1+\lambda) \cdot \left(x - \frac{3\lambda}{1+\lambda}\right)^2 + \left(1 + 8\lambda - \frac{9\lambda^2}{1+\lambda}\right)$$



Since all $L(x, \lambda)$ passes through the point $(2, 5)$, we must have $p^* \geq \inf_x L(x, \lambda)$ for all $\lambda \geq 0$.

The dual objective function is given by
$$g(\lambda) = 1 + 8\lambda - \frac{9\lambda^2}{1+\lambda}$$
$$= \frac{1 + 9\lambda - \lambda^2}{1+\lambda}$$
$$= (-\lambda + 10) - \frac{9}{1+\lambda}$$

$= (-\lambda+10) - \frac{9}{1+\lambda}$



(c) Since $(10-\lambda)$ is linear, and $\frac{-9}{1+\lambda}$ is concave, hence $g(\lambda)$ is a concave function of λ.

To find the dual optimal value,

$$g'(\lambda) = -1 + \frac{9}{(1+\lambda)^2} = 0$$

$$\lambda^* = 2$$

$$\Rightarrow \quad g(\lambda^*) = 10 - \lambda^* - \frac{9}{1+\lambda^*} = 5$$

Strong duality does hold!

(d) For each $u \succeq 0$, the feasible set now becomes

$$x^2 - 6x + (8-u) \leq 0$$

$$\Rightarrow \quad (x-3)^2 \leq 1+u$$

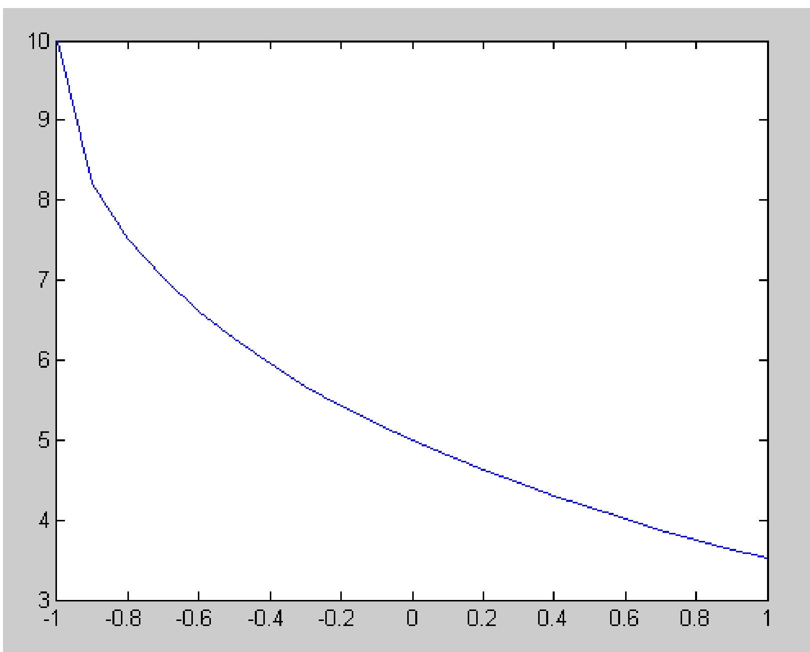
$$3-\sqrt{1+u} \leq x \leq 3+\sqrt{1+u}$$

Hence, the optimal value

$$p^*(u) = \left(3-\sqrt{1+u}\right)^2 + 1$$

$$= 11+u - 6\sqrt{1+u}$$

$$\frac{d\,p^*(0)}{du} = 1 - \frac{3}{\sqrt{1+u}}\Bigg|_{u=0} = -2 = -\lambda^*$$



5.21 (a) This is indeed a convex problem.

The optimal value is $e^{-0} = 1$

(b) $L(x, \lambda) = e^{-x} + \lambda\left(\frac{x^2}{y} - 0\right)$, $\lambda \geq 0$

$g(\lambda) = \min_{x,\, y>0} L(x, \lambda)$

Clearly $y \to +\infty$ should minimize the second term, then $x \to -\infty$ minimizes the first term.

Hence $g(\lambda) = 0 \quad \forall\ \lambda \geq 0$

The optimal solution λ^* can by any nonnegative real number, and $d^* = 0$

The duality gap is $1 - 0 = 1$

(c) Slater condition does not hold.

The only point in the domain D that satisfies the constraint is $x = 0$.

(d) $p^*(u) = +\infty$ if $u < 0$

If $u > 0$ as $y \to +\infty$, $x = \sqrt{yu}$, we have $e^{-x} \to 0$

$\therefore\ p^*(u) = 0$ if $u > 0$. Also, we knew that $p^*(0) = 1$.

The inequality

$p^*(u) \geq p^*(0) - \lambda^* u$

$\Leftrightarrow p^*(u) \geq 1 - \lambda^* u$ does not hold

for any $\lambda^* \geq 0$.

5.27. The KKT condition is

① $Gx = h$

② $\nu \in R^p$

③ x min $\|Ax - b\|^2 + \nu^T(Gx - h)$

From ③, we must have

$$2A^TAx - 2A^Tb + G^T\nu = 0$$

$$\therefore \quad x = \frac{1}{2}(A^TA)^{-1}[2A^Tb - G^T\nu]$$

From ①,

$$Gx = \frac{1}{2}G(A^TA)^{-1}[2A^Tb - G^T\nu] = h$$

$$\Rightarrow \frac{1}{2}G(A^TA)^{-1}G^T\nu = G(A^TA)^{-1}A^Tb - \text{[redacted]}\, h$$

$$\Rightarrow \nu = (\text{[redacted]}\, 2G(A^TA)^{-1}G^T)^{-1}[G(A^TA)^{-1}A^Tb - \text{[redacted]}\, h].$$

5.28. The KKT condition is

① $\vec{x} \in R^4$

② $\lambda_i \geq 0 \quad \forall i$

③ $\lambda_i (Ax-b)_i = 0 \quad i = 1, 2, 3, 4, 5$

④ x minimizes $L(x, \lambda) = c^T x + \lambda^T (Ax - b)$

Note: This is a tricky question because you are not supposed to use any LP solver. However, since we are already given x^*, it should be easier to check.

We first verify that there exists λ^*, such that λ^* and $x^* = (1,1,1,1)^T$ satisfy the KKT condition.

To see this, note that

$$Ax - b = \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 3 \end{bmatrix} - \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This is an important step that many students missed. Since the equation below $A^T\lambda = -c^T$ has 5 unknowns and 4 equations, you won't be able to find the right λ unless you know $\lambda_5 = 0$.

From ③, $\lambda_5 = 0$.

From ④, in order that x^* minimize $L(x, \lambda)$, we must have $A^T \lambda = -c^T$, hence

$$\begin{bmatrix} -1 & -1 & 0 & -6 & x \\ -6 & -2 & 3 & -11 & x \\ 1 & 7 & -10 & -2 & x \\ 3 & 1 & -1 & 12 & x \end{bmatrix} \lambda = -\begin{bmatrix} 47 \\ 93 \\ 17 \\ -93 \end{bmatrix} \Rightarrow \lambda^* = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 7 \\ 0 \end{bmatrix}$$

not important since $\lambda_5 = 0$

Clearly λ^* and x^* satisfy all KKT conditions.

To show x^* is unique, we need to show that no other x and satisfy the KKT condition with λ^*. Since $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \neq 0$, from ③, we have

$$(Ax - b)_i = 0 \quad i = 1, 2, 3, 4$$

$$\Rightarrow \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \end{bmatrix} x = \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, x^* is the unique solution.

Note: In general, a convex problem may not have a unique solution. This is why we have to do this additional check using λ^*.