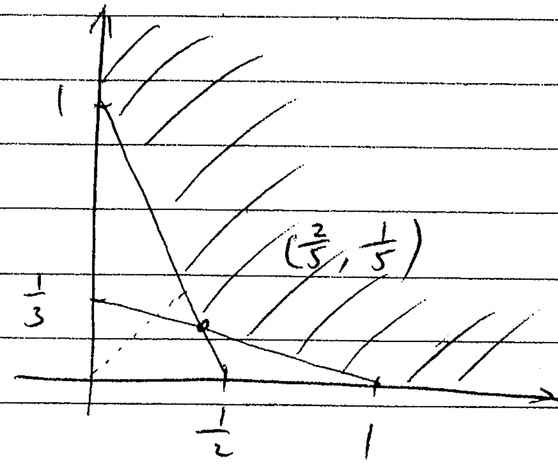
4/7

4.1. The feasible set is



The optimal point and optimal value is

a) $\left(\frac{2}{5}, \frac{1}{5}\right) \Rightarrow \frac{3}{5}$

b) ϕ $\quad -\infty$

c) $(0, 1) \Rightarrow 0$

d) $\left(\frac{1}{3}, \frac{1}{3}\right) \Rightarrow \frac{1}{3}$

can be easily verified by the normal cone.

e) The minimum value may occur when

$$x_1^2 + 9x_2^2 = r$$

touches the line $x_1 + 3x_2 = 1$ at some point (a, b)

we thus have

$$\begin{cases} \frac{2a}{1} = \frac{18b}{3} \\ a + 3b = 1 \end{cases}$$

$$\Rightarrow a = \frac{1}{2}, \quad b = \frac{1}{6}$$

This point is infact at the boundary of the feasible set

$\Rightarrow$ optimal value is $a^2 + 9b^2 = \frac{1}{2}$

4.3. Let $f(x) = \frac{1}{2} x^T P x + q^T x + r$

$\nabla f(x) = Px + q^T$

At the point $x^* = (1, \frac{1}{2}, -1)$

$$\nabla f(x^*) = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

For and $x \in [-1, 1]^3$, we have

$\nabla f(x^*)^T (x - x^*)$

$$= \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - \frac{1}{2} \\ x_3 + 1 \end{bmatrix} = -(x_1 - 1) + 2(x_3 + 1) \geq 0$$

$\therefore$ x^* is the optimal solution.

Or, the normal cone at x^* is given by

$$N_C(x^*) = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \alpha \geq 0, \beta \geq 0 \right\}$$

Thus, $-\nabla f(x^*) \in N_C(x^*)$.

HW2 solution (cont.)

Wednesday, February 8, 2023 9:26 AM

4.7(b)

Start with the original problem

$$\min \quad \frac{f_0(x)}{c^T x + d}$$

$$\text{sub to} \quad f_i(x) \leq 0$$
$$Ax = b$$
$$c^T x + d > 0.$$

Similar to linear-fractional program, let

$$y = \frac{x}{c^T x + d}, \qquad t = \frac{1}{c^T x + d}$$

Then there is a 1:1 mapping between

$$x,\ c^T x + d > 0 \iff (y, t),\ \ c^T y + dt = 1,\ \ t > 0$$

Now, objective func. becomes

$$\frac{f_0(x)}{c^T x + d} = t \cdot f_0\left(\frac{y}{t}\right)$$

which is convex

The constraints become

$$t f_i\left(\frac{y}{t}\right) \leq 0$$
$$Ay = bt$$
$$c^T y + dt = 1$$

which are convex #

To formally show that they are equivalent, Let (A) and (B) denote the original and new problem, respectively.

We just need to show that (i) for the optimal solution x^* to (A), there exist (y, t) for (B) that is feasible and attain the same value; and (ii) for the optimal solution (y^*, t^*) to (B), there exits x for (A) that is feasible and attain the same objective value.

Both directions can be verified easily with the mapping

$$y = \frac{x^*}{c^T x^* + d} \qquad t = \frac{1}{c^T x^* + d}$$

and $$x = y^* / t^*$$

4.8 (d)

Minimizing a linear func. over the prob. simplex

$$\begin{aligned} \min \quad & c^T x \\ \text{sub to} \quad & \mathbf{1}^T x = 1, \quad x \geq 0 \end{aligned}$$

The solution will be such that the element of x_i with the smallest value of c_i will be 1, and all other elements will be zero.

If the constraint is replaced by

$$\mathbf{1}^T x \leq 1 \quad , x \geq 0$$

then the above solution is still the optimal if there exists at least one c_i that is < 0.
Otherwise, i.e., if all c_i's are non-negative, the solution is all 0's.

4.13

- Focus on the constraint

$$Ax \le b \quad \text{for all } A \in \mathcal{A}$$

where $\mathcal{A} = \{A \mid \bar{A}_{ij} - V_{ij} \le A_{ij} \le \bar{A}_{ij} + V_{ij}, \ \forall i,j\}$

- Thus, we need

$$\sum_j A_{ij} x_j \le b \quad \text{for all } A_{ij} \text{ such that}$$

$$\bar{A}_{ij} - V_{ij} \le A_{ij} \le \bar{A}_{ij} + V_{ij}$$

Given x_j, this is equivalent to

$$\sum_j \bar{A}_{ij} x_j + \sum_j |V_{ij} x_j| \le b \qquad (*)$$

We can then min $c^T x$, sub to $(*)$

- Using $u_{ij} = |V_{ij} x_j|$, we can write the LP as

$$\min \ c^T x$$

$$\text{sub to} \quad -u_{ij} \le V_{ij} x_j \le u_{ij} \quad \forall i,j$$

$$\sum_j \bar{A}_{ij} x_j + \sum_j u_{ij} \le b \quad \forall i$$

- Note that for any optimal solution to the LP, there always exist an optimal solution with

$$u_{ij} = |V_{ij} x_j|$$

$\Rightarrow$ Hence, the LP is equivalent to the original problem. #

4.21 (b)

Minimizing a linear func. over an ellipsoid

$$\min \quad c^T x$$
$$\text{sub to} \quad (x-x_c)^T A (x-x_c) \le 1$$

Since the obj. is linear, there always exists a solution $\bar{x}$ at the boundary.

The normal vector at $\bar{x}$ is $A(\bar{x}-x_c)$

Hence, the optimality condition leads to

$$-c = \lambda A(\bar{x}-x_c) \text{ for } \lambda \ge 0$$

$$\Rightarrow \quad \bar{x} = -\lambda A^{-1} c + x_c.$$

Substituting this into

$$(\bar{x}-x_c) A (\bar{x}-x_c) = 1,$$

we have

$$\lambda^2 c^T A^{-1} A A^{-1} c = 1$$

$$\lambda = \frac{1}{\sqrt{c^T A^{-1} c}}$$

Hence,

$$\bar{x} = -\frac{A^{-1} c}{\sqrt{c^T A^{-1} c}} + x_c. \qquad \#$$

4.58

The problem can be formulated as

$$\max \sum_{t=1}^{T} \beta^t u(c_t)$$

$$\text{sub to} \quad c_t \ge 0$$
$$k_{t+1} = k_t + f(k_t) - c_t \quad \forall t \ge 0$$
$$k_t \ge 0 \quad \forall t \ge 1$$

When $u(\cdot)$ is increasing, this is equivalent to

$$\max \sum_{t=1}^{T} \beta^t u(c_t)$$

$$\text{sub to} \quad c_t \geq 0$$

$$k_{t+1} \leq k_t + f(k_t) - c_t \quad \forall t \geq 0 \quad (**)$$

$$k_t \geq 0 .$$

The reason they are equivalent is because we can always increase c_t to make $(**)$ an equality, and by doing that we only increase the objective function value.

This is a convex problem because $u(\cdot)$ & $f(\cdot)$ are concave.