2. "$\Rightarrow$" direction: If $f$ is convex then

$$f(y) \geq f(x) + f'(x)(y-x). \qquad (*)$$

To see this, let

$$g(t) = \frac{f(x+t(y-x)) - f(x)}{t \cdot (y-x)}, \quad 0 < t \leq 1$$

Note that as $t \to 0$,

$$\lim_{t \to 0} g(t) \to f'(x)$$

To show $(*)$, it suffices to show that

$$g(t) \leq g(1) \quad \text{for all } t \leq 1$$

$$\Leftrightarrow \quad \frac{f(x+t(y-x)) - f(x)}{t \cdot (y-x)} \leq \frac{f(y) - f(x)}{y-x}$$

$$\Leftrightarrow \quad f((1-t)x + ty) \leq (1-t)f(x) + t f(y),$$

which follows from the convexity of $f$.

"$\Leftarrow$" direction: If $(*)$ holds, then $f$ is convex.

For any $x, y$, let $z = \theta x + (1-\theta) y$.

Then
$$f(x) \geq f(z) + f'(z)(x-z)$$
$$f(y) \geq f(z) + f'(z)(y-z)$$

Multiplying by $\theta$ & $(1-\theta)$, respectively, and summing up, the result follows

3. (a) The delay of an M/M/1 queueing system is given by

$$E[D] = \frac{\frac{1}{\mu}}{1-\frac{\lambda}{\mu}}.$$

For fixed $d$, the set

$$\left\{\lambda \,\middle|\, \frac{\frac{1}{\mu}}{1-\frac{\lambda}{\mu}} \le d\right\}$$

is convex because the function $\frac{\frac{1}{\mu}}{1-\frac{\lambda}{\mu}}$ is a convex function.

If $d$ is also a variable, the set

$$\left\{(\lambda, d) \,\middle|\, \frac{\frac{1}{\mu}}{1-\frac{\lambda}{\mu}} \le d\right\}$$

is still convex since the function

$$\frac{\frac{1}{\mu}}{1-\frac{\lambda}{\mu}} - d$$ is a convex function.

(b) The constraint can be written as

$$\log\left(1 + \frac{p_0}{\sum_{i=1}^{I} p_i + N}\right) \ge r$$

If $r$ is fixed, this is equivalent to

$$p_0 + \sum_{i=1}^{7} p_i + N \geq e^r \left( \sum_{i=1}^{7} p_i + N \right)$$

The set of vectors $[p_0\ p_1 \cdots p_7]$ such that the above inequality is satisfied is convex because the function

$$-\left(p_0 + \sum_{i=1}^{7} p_i + N\right) + e^r \left( \sum_{i=1}^{7} p_i + N \right)$$

is a linear (and thus convex) function.

However, if $r$ is also a variable, then we cannot draw this conclusion

(c) The constraint is

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 \leq r^2 \Leftrightarrow \frac{1}{2}\sqrt{x^2+y^2} - r \leq 0$$

Since the function $\sqrt{\frac{x^2}{4} + \frac{y^2}{4}} - r$ is convex, both the set of $(x, y)$ for a given $r$, or the set of $(x, y, r)$ that satisfy the constraint is convex.

Prob 3.5. We will show that $F''(x) \geq 0$. Note that

$$F''(x) = \frac{2}{x^3}\int_0^x f(t)\,dt - \frac{2}{x^2} f(x) + \frac{1}{x} f'(x). \qquad (*)$$

Since $f(x)$ is convex,

$$f(t) \geq f(x) + f'(x)(t-x)$$

Integrate over $t \in (0,x)$, we have

$$\int_0^x f(t)\,dt \geq x f(x) - \frac{x^2}{2} f'(x)$$

Substitute into $(*)$, we have

$$F''(x) = \frac{2}{x^3}\left[\int_0^x f(t)\,dt - x f(x)\right] + \frac{1}{x} f'(x)$$
$$\geq -\frac{1}{x} f'(x) + \frac{1}{x} f'(x) = 0.$$

Prob 3.15 (b). Concavity and monotonicity are trivial since $0 \leq \alpha \leq 1$. Also

$$u_\alpha(1) = \frac{1^\alpha - 1}{\alpha} = 0.$$

Prob. 3.16

a) convex

b) neither convex nor concave

c) convex since the Hessian is positive semi-definite

d) neither convex nor concave

e) convex since the Hessian is positive semi-definite

f) concave since the Hessian is negative semi-definite

Prob 3.21

a) Since $\|A^{(i)}x - b^{(i)}\|$ is convex for each $i$,
and $f(x)$ is the piecewise maximum of convex functions,
$f(x)$ is also convex

b) $f(x) = \max_{1 \le k_1 < k_2 < k_3 < \cdots < k_r \le n} \left\{ \sum_{i=1}^{r} |x_{k_i}| \right\}$

Since $\sum_{i=1}^{r} |x_{k_i}|$ is convex for each set of $k_1, \ldots, k_r$,
$f(x)$ is the piecewise maximum of convex functions,
$\therefore$ $f(x)$ is also convex.

3.22.

a) $f(x) = -\log g(x)$, $g(x) = -\log \left( \sum_{i=1}^{m} e^{a_i^T x + b_i} \right)$

$\mathrm{dom}\, f = \{ x \mid \sum_{i=1}^{m} e^{a_i^T x + b_i} < 1 \}$ is convex.
Further, $-\log x$ is convex and non increasing
$g(x)$ is concave
Hence, $f(x)$ is convex

b)

$f(x, u, v) = -\sqrt{g_1(u)\, g_2(u, v, x)}$,
where $g_1(u) = u$
$g_2(u, v, x) = v - \frac{x^T x}{u}$

$\mathrm{dom}\, f = \{ (x, u, v) \mid uv > x^T x,\ u, v > 0 \}$

$= \{ (x, u, v) \mid u > \frac{x^T x}{v},\ u, v > 0 \}$

is convex since $\frac{x^T x}{v} - u$ is convex.
Further, $-\sqrt{x_1 x_2}$ is convex and component-wise non-increasing
$g_1(u)$ is concave, $g_2(u, v, x)$ is concave
$\therefore$ $f(x, u, v)$ is convex.

3.22 c) $f(x,u,v) = -\log u - \log\left(v - \frac{x^Tx}{u}\right)$

$$= -\log u - \log(g(x,u,v))$$

where $g(x,u,v) = v - \frac{x^Tx}{u}$ is concave

Further $-\log x$ is convex and non-increasing

Hence, $f(x,u,v)$ is convex

d) $f(x,t) = -t\left(\cancel{1} - \frac{\|x\|_p^p}{t^p}\right)^{1/p}$

$$= -t^{1-1/p}\left(t - \frac{\|x\|_p^p}{t^{p-1}}\right)^{1/p}$$

$$= -g_1(t)^{1-1/p} \cdot g_2(x,t)^{1/p}$$

where

$$g_1(t) = t$$
$$g_2(t) = t - \frac{\|x\|_p^p}{t^{p-1}}$$

$\mathrm{dom}\, f = \{(x,t) \,/\, t \geq \|x\|_p\}$ is a convex set since $\|x\|_p - t$ is a convex function

Further $-x^{1-1/p}y^{1/p}$ is convex and componentwise non-increasing, $g_1(t)$ is concave and $g_2(t)$ is concave.

Hence, $f(x,t)$ is convex

e) Similar to c) & d).