Solution - HW1 Convex Sets

1. First, we can show that aff C as defined is an affine set. To see this, for any finite set of points in aff C, we can write them as

$$y_1 = \theta_1^1 x_1 + \cdots + \theta_k^1 x_k$$
$$y_2 = \theta_1^2 x_1 + \cdots + \theta_k^2 x_k$$
$$\vdots$$
$$y_m = \theta_1^m x_1 + \cdots + \theta_k^m x_k$$

where $x_1, \ldots x_k \in C$, and

$$\sum_{i=1}^{k} \theta_i^j = 1, \quad j = 1, \ldots, m.$$

Hence, for any $\theta_1, \ldots \theta_m$ such that $\sum_{j=1}^{m} \theta_j = 1$, let $y = \sum_{j=1}^{m} \theta_j y_j$. Then

$$y = x_1 \sum_{j=1}^{m} \theta_j \theta_1^j + x_2 \cdot \sum_{j=1}^{m} \theta_j \theta_2^j + \cdots + x_k \sum_{j=1}^{m} \theta_j \theta_k^j$$

and the coefficients satisfy

$$\sum_{j=1}^{m} \theta_j \theta_1^j + \cdots + \sum_{j=1}^{m} \theta_j \theta_k^j$$
$$= \sum_{j=1}^{m} \theta_j \left[\theta_1^j + \cdots + \theta_k^j \right] = \sum_{j=1}^{m} \theta_j = 1$$

Hence $y \in$ aff C. This proves that aff C is an affine set.

Finally, for any affine set that contains C, by definition such set must contain $x_1, \ldots x_k \in C$, and therefore must contain $\theta_1 x_1 + \cdots + \theta_k x_k$, for all $x_1, \ldots x_k \in C$ and $\theta_1 + \cdots + \theta_k = 1$. Hence, aff C as defined is the smallest affine set containing C.

2. The affine hull is the plane that goes through the three points, which is given by

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$$

The ~~aff~~ convex hull is the triangle with the three points as vertices. which is given by

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1, \quad x \geq 0, \; y \geq 0, \; z \geq 0$$

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Let C_i, $i \in A$ by a collections of convex sets. We now show that $C = \bigcap_{i \in A} C_i$ is also a convex set.

For any points $x_1, \cdots x_k \in C$, $\theta_1, \cdots \theta_k \geq 0$, $\theta_1 + \cdots \theta_k = 1$ we have

$$x_1, \cdots x_k \in C_i \quad \text{for all } i \in A$$

Hence

$$\theta_1 x_1 + \cdots + \theta_k x_k \in C_i \quad \text{for all } i \in A$$

Therefore

$$\theta_1 x_1 + \cdots + \theta_k x_i \in \bigcap_{i \in A} C_i = C. \qquad \#$$

2.11. Let $y_1, \dots, y_n > 0$ and $\prod_{i=1}^{n} y_i \geq 1$

$z_1, \dots, z_n > 0$ and $\prod_{i=1}^{n} z_n \geq 1$.

Let $h_i = \theta y_i + (1-\theta) z_i$, $i = 1, \dots, n$

for some $\theta \in (0, 1)$.

We need to show that

$$\prod_{i=1}^{n} (\theta y_i + (1-\theta) z_i) \geq 1$$

Using the inequality in the Hint,

$$\prod_{i=1}^{n} (\theta y_i + (1-\theta) z_i)$$

$$\geq \prod_{i=1}^{n} y_i^{\theta} z_i^{(1-\theta)}$$

$$= \left(\prod_{i=1}^{n} y_i\right)^{\theta} \cdot \left(\prod_{i=1}^{n} z_i\right)^{1-\theta} \geq 1 \qquad \#$$

2.12

a) Yes. Intersection of half-spaces

b) Yes. same reason

c) Yes. same reason

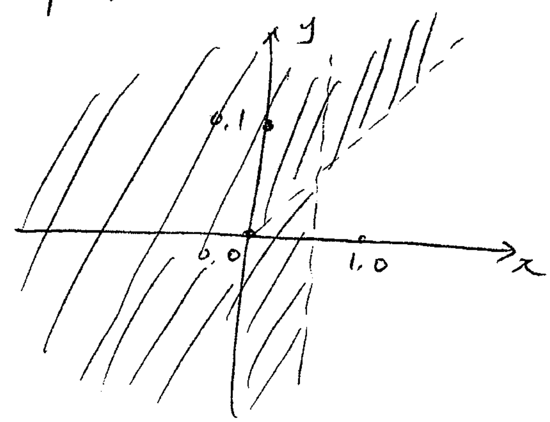
d) Yes. The set can be writen as

$$\bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$

each of which is a half-space.

e) No. Counter example, $S = \{(0,0), (0,1)\}$

$T = \{(1, 0)\}$



f) Yes. The set can be written as

$$\bigcap_{y \in S_2} \{x \mid x+y \in S_1\}$$

$$= \bigcap_{y \in S_2} (S_1 - y)$$

g) Yes. $\|x-a\|_2^2 \le \theta^2 \|x-b\|_2^2$

$$\Leftrightarrow \|x\|_2^2 - 2a\cdot x + \|a\|^2 \le \theta^2 \|x\|_2^2 - 2\theta^2 b\cdot x + \theta^2 \|b\|_2^2$$

$$\Leftrightarrow (1-\theta^2)\|x\|_2^2 - 2(a-\theta^2 b)x + (\|a\|^2 - \theta^2 \|b\|_2^2) \le 0$$

we can then verify by definition.

(or, note that the inequality defines a ball.)

Solution . – Convex Functions

1. To show that $\|x\|$ is convex, note that for any x, y and $0 \le \theta \le 1$

$$
\begin{aligned}
& \|\theta x + (1-\theta) y\| \\
\le\ & \|\theta x\| + \|(1-\theta) y\| && (\text{by iv}) \\
=\ & \theta \|x\| + (1-\theta) \|y\| && (\text{by iii})
\end{aligned}
$$