Lec9

Friday, January 27, 2023 6:56 PM

Basic properties of convex problems

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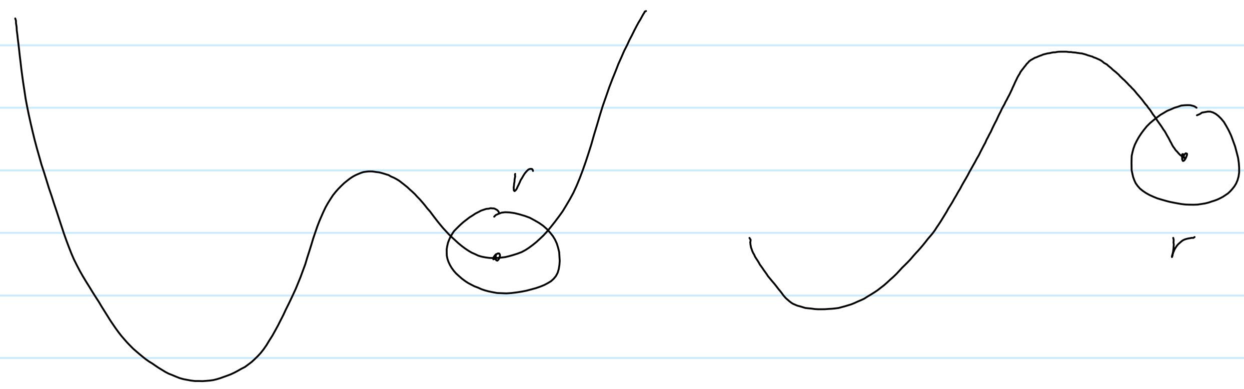
① Every local optimum is also global optimum.

- x is locally optimal if x is feasible and

$$f_0(x) = \inf \{ f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \le r \}$$

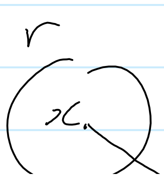
for some $r > 0$

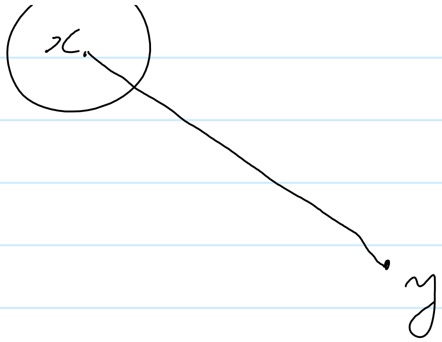


optimal within a neighborhood r.

could also occur at the boundary of the feasible set.

Proof: If $f(y) < f(x)$ for another feasible point y, then the line segment xy must lie in the feasible set, which is convex





By convexity of f, for any point

$$z = \theta x + (1-\theta) y \qquad 0 < \theta < 1$$

we must have

$$f(z) \le \theta f(x) + (1-\theta) f(y) < f(x)$$

Hence, there must exist a point in the neighborhood of x that has a smaller function value than $f(x)$.

$\Rightarrow$ A contradiction.

Note that the same conclusion holds for the looser definition of convex problems as well.

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(2) Necessary conditions for optimality are also sufficient.

More details soon.

Maximize a convex function

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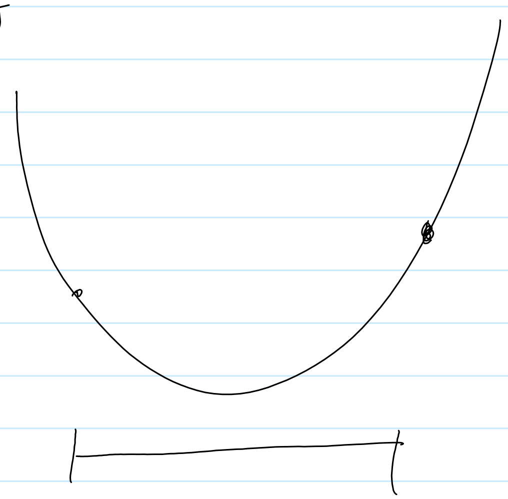
Ⓠ What if we maximize a convex function on a convex set?

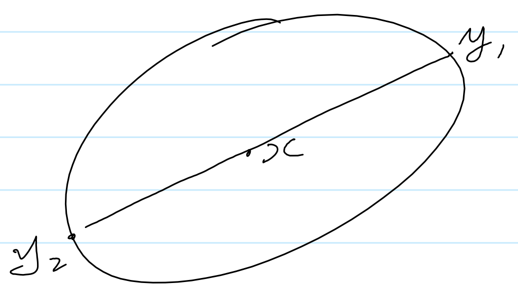
Ⓐ The maximum will be attained at the boundary



Both could be local optimums!

(Not always)

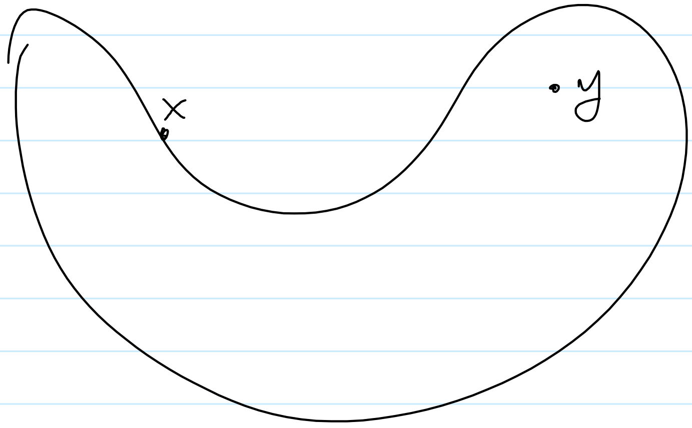
- Let f be a convex function $C \to \mathbb{R}$.
  C is a convex set.
  Let D be a closed bounded set contained in C.
  Then for any $x \in \text{int } D$, there exists a $y \in \text{bd } D$ such that $f(y) \geq f(x)$



$f(x) \leq \max\{f(y_1), f(y_2)\}$.

$\Rightarrow$ Maximum on a closed set D must be attained at the boundary.

Similarly, if you min a convex function on a non-convex set, then a local minimum at the boundary may not be global minimum



Conditions for optimality

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When is a point $\bar{x}$ optimal?

- From calculus, a common condition is

$$\nabla f(\bar{x}) = 0$$

- However, this may only work when there are no constraints.

- Our goal is to set up the following necessary and sufficient condition:

  - Assume that f is convex and differentiable, and C is a convex set.
  - A point $\bar{x}$ is the global minimum of f in C if and only if

$$\nabla f(\bar{x})^T (x - \bar{x}) \geq 0 \qquad (*)$$

  for $x \in C$.

- Note that if $\bar{x}$ is in the interior of C, then $(*)$ is possible only when

$$\nabla f(\bar{x}) = 0$$

  - We will look at $(*)$ for the case when $\bar{x}$ is at the boundary of C later.

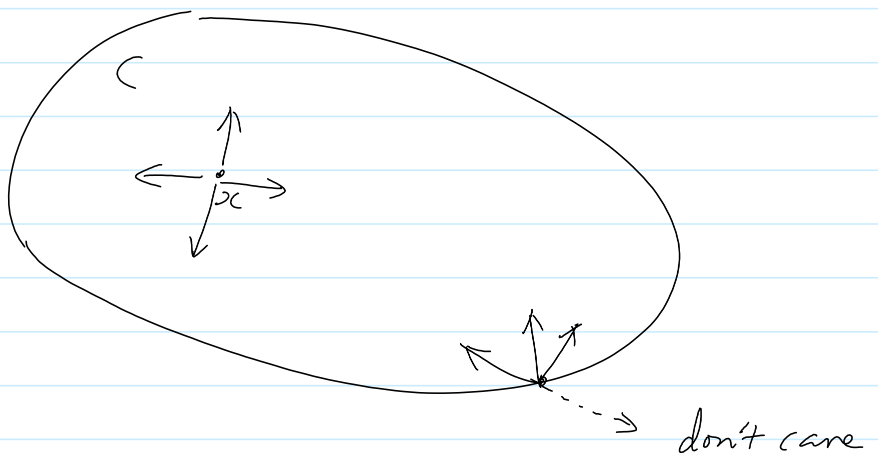
- However, sometimes we want to work with functions that are not even differentiable!

  - We will give a more general version below,

- We will give a more general version below, which will lead to (*).

Necessary condition:

- Let us look at the necessary condition first
- If $\bar{x}$ a local optimal of f in C, then...
  - f may not even be convex.
- Roughly speaking, f must be non-decreasing in any direction pointing towards the interior of the feasible set C.



- Define the directional derivative of a function f at $\bar{x}$ in a direction $d \in E$ as

$$f'(\bar{x}; d) = \lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t},$$

when this limit exists.

- Note that if f is differentiable with gradient $\nabla f(x)$, then

$$f'(\bar{x}, d) = \frac{d}{dt} f(\bar{x} + td)\Big|_{t=0} = [\nabla f(\bar{x})]^T \cdot d.$$

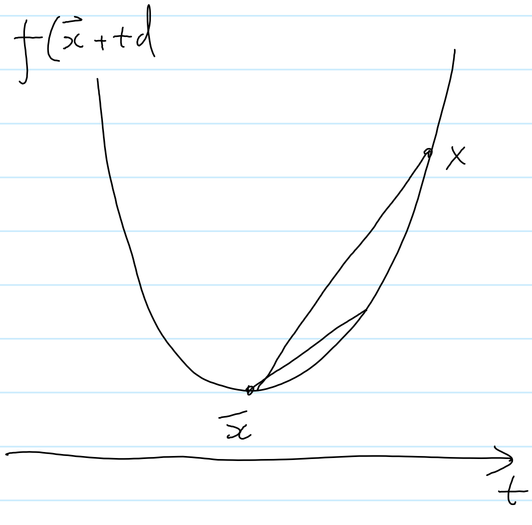
- The nice part of $f'(\bar{x}, d)$, however, is that it may exist even if f is not differentiable.

- In fact, it always exist for convex (even non-differentiable) functions.

Existence of directional derivatives for convex functions.

Suppose that the set $C \subset E$ is convex, and that the function $f: C \to R$ is convex. Then for any points $\bar{x}$ and x in C, the directional derivative $f'(\bar{x}; x - \bar{x})$ always exists in $[-\infty, +\infty)$

- could be minus-infinite



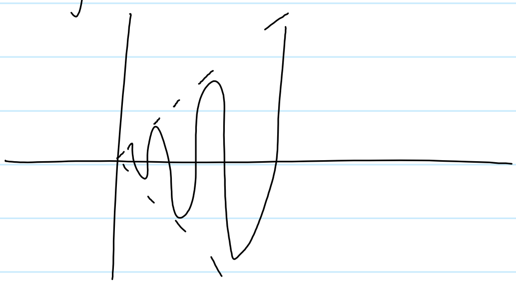
$$\frac{f(\bar{x}+td) - f(\bar{x})}{t}$$

is decreasing as $t \downarrow 0$.

$$f(\bar{x}; x-\bar{x}) = -\infty$$

– May NOT exist for arbitrary non-convex functions

$$f(x) = x \sin\frac{1}{x}$$



A necessary condition (based on directional derivative):

Suppose that C is a convex set in E and that the point $\bar{x}$ is a local minimizer of $f: C \to R$. Then for any point x in C, the direction derivative, if it exists, satisfies

$$f'(\bar{x}; x-\bar{x}) \geq 0$$

Proof: If $f'(\bar{x}; x-\bar{x}) < 0$ for some $x \in C$, then the function value will be decreasing in the direction of $\bar{x} \to x$ for a small interval.

$\Rightarrow$ Contradiction.

Note:

– This necessary condition holds for any functions

- This necessary condition holds for any functions
- For convex problem, the directional derivative always exists.

$$\Rightarrow \quad f'(\bar{x}; x-\bar{x}) \geq 0 \quad \text{for all } x \in C$$

must always hold.

Necessary conditions for differentiable functions

If in addition f is differentiable,

$$f'(\bar{x}, x-\bar{x}) = [\nabla f(\bar{x})]^T (x-\bar{x}) \geq 0 \text{ for all } x \in C$$

- This is exactly $(*)$

Sufficient conditions

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First-order <u>sufficient</u> condition:

- Suppose that the set $C \subset E$ is convex and that the function $f: C \to \mathbb{R}$ is convex.

  Let $\bar{x} \in C$. If the condition $f'(\bar{x}; x-\bar{x}) \geq 0$ holds for all $x \in C$, then $\bar{x}$ is a global minimizer of f on C.

  - If in addition f is differentiable, then the sufficient conditions becomes

$$[\nabla f(\bar{x})]^T (x-\bar{x}) \geq 0 \quad \text{for all } x \in C.$$

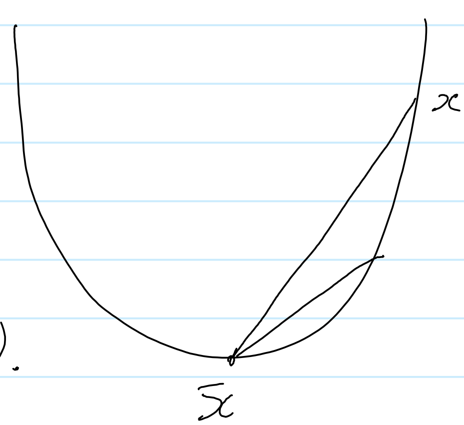
- Same as the necessary condition!
- Note that in general, this statement does not hold for non-convex problems.

Proof: Again, we can show that

$$\frac{f(\bar{x}+t(x-\bar{x}))-f(\bar{x})}{t}$$

is decreasing as $t \downarrow 0$.

The limit as $t \downarrow 0$ is $f'(\bar{x}; x-\bar{x})$.



Hence,

$$f(\bar{x} + t(x-\bar{x})) - f(\bar{x})$$
$$\geq t \cdot f'(\bar{x}; x-\bar{x}) \geq 0.$$

Let $t=1$. We have $f(\bar{x}) \leq f(x)$.

The result then follows.

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Optimality at the boundary

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– Recall that if the set $C \subset E$ is convex and the function $f : C \to R$ is convex and diffentiable,

Then a necessary & sufficient condition for $\bar{x} \in C$ to be a global minimizer of f on C is

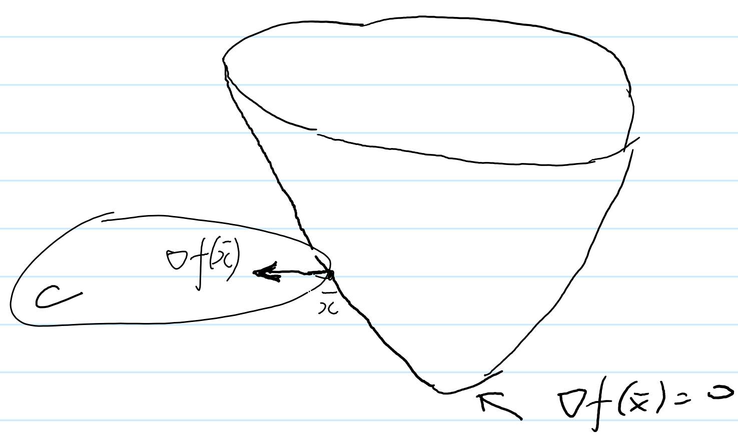
$$[\nabla f(\bar{x})]^T (x - \bar{x}) \geq 0 \quad \text{for all } x \in C.$$

We have the following

(a) if $\bar{x}$ belongs to the interior of C, then

$$\nabla f(\bar{x}) = 0$$

(b) if $\bar{x}$ is at the boundary.



$\Rightarrow$ $-\nabla f(\bar{x})$ must point outwards of C.

– Define the normal cone $N_C(\bar{x})$ to a convex set C at point $\bar{x}$

$$N_C(\bar{x}) = \{ d \mid \langle d, x - \bar{x} \rangle \leq 0 \quad \text{for all } x \in C \}$$

$N_C(\bar{x}) = \{ d \mid \langle d, x - \bar{x} \rangle \le 0 \text{ for all } x \in C \}$

- This is the set of "outward" directions.

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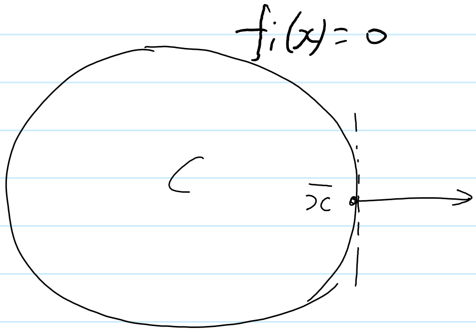
① Smooth surface



$N_C(\bar{x})$ containing one normal vector

- If the constraint is defined by $f_i(x) \le 0$,
  - At a point $\bar{x}$ at the boundary, we have

$$f_i(\bar{x}) = 0$$

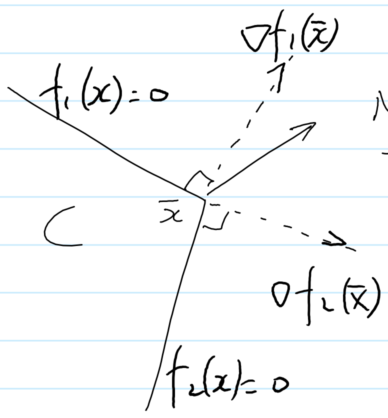
$$\Rightarrow \text{For all } x \in C$$

$$(\nabla f_i(\bar{x}))^T (x - \bar{x}) \le f(x) - f(\bar{x}) \le 0$$

for all $x \in C$

$\Rightarrow \nabla f_i(\bar{x})$ is the normal vector!

② At a Corner



$N_C(\bar{x})$ is a cone that contains all normal vectors (when $\bar{x}$ is tight for multiple constraints)

- The boundary of the cone is the two normal vectors $\nabla f_1(\bar{x})$ and $\nabla f_2(\bar{x})$ for the two respective constraints.

vectors $\nabla f_1(\bar{x})$ and $\nabla f_2(\bar{x})$ for the two respective constraints.

- $N_C(\bar{x})$ then contains all vectors of the form

$$\underset{\substack{\vee\!\!\mid\\0}}{\lambda_1} \nabla f_1(\bar{x}) + \underset{\substack{\vee\!\!\mid\\0}}{\lambda_2} \nabla f_2(\bar{x})$$ "conic combinations"

Then, our necessary & sufficient condition

$$f'(\bar{x}; x-\bar{x}) = \left[\nabla f(\bar{x})\right]^T (x-\bar{x}) \geq 0$$
for all $x \in C$

$\Leftrightarrow \quad \langle -\nabla f(\bar{x}), x-\bar{x} \rangle \leq 0$ for all $x \in C$

$\Leftrightarrow \quad -\nabla f(\bar{x}) \in N_C(\bar{x})$.

Negative gradient must belong to the normal cone ("outwards")

- If there is only 1 normal vector $d = \nabla f_i(\bar{x})$, then

$$-\nabla f(\bar{x}) = \lambda \cdot d \quad \text{for some } \lambda \geq 0$$

- If $N_C(\bar{x})$ contains the conic combinations of multiple vectors, e.g. $\nabla f_1(\bar{x})$ & $\nabla f_2(\bar{x})$ then

$$-\nabla f(\bar{x}) = \underset{\substack{\vee\!\!\mid\\0}}{\lambda_1} \nabla f_1(\bar{x}) + \underset{\substack{\vee\!\!\mid\\0}}{\lambda_2} \nabla f_2(\bar{x})$$

- Note that constraints that are not binding play no roles.

Note: If we define $N_C(\bar{x}) = \{0\}$ when $\bar{x}$ lies in the interior of C, then (a) becomes a special case of (b).

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