Lec8-mwf

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Vector function composition

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If $g: R^n \to R$, $h: R \to R$, take a restriction to a line

$$f(\lambda) = h(g(x+\lambda v))$$

g convex/concave in x

$\Leftrightarrow g(x+\lambda v)$ convex/concave in λ

$\Rightarrow$ Exactly the same rule apply!

$$f(x) = h(g(x)) \quad : \quad g = [g_1 \cdots g_K]^T$$

Assume without loss of generality that $x \in R$ (e.g. restricted the function to a line).

$h: R^K \to R \qquad h(y_1, \cdots, y_K)$

$g_i: R \to R \qquad i = 1, 2, \cdots, K$

$$g = [g_1, g_2, \cdots, g_K]^T$$

$$f(x) = h(g_1(x), g_2(x), \cdots g_K(x))$$

Then

$$f'(x) = \nabla h(g(x))^T \cdot g'(x) \qquad \in R$$

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) \cdot g'(x) + \nabla h(g(x))^T g''(x) \qquad \in R$$

where

$$\nabla h = \begin{bmatrix} \frac{\partial h}{\partial y_1} \\ \vdots \\ \frac{\partial h}{\partial y_K} \end{bmatrix} \qquad g' = \begin{bmatrix} g_1' \\ \vdots \\ g_K' \end{bmatrix} \qquad g'' = \begin{bmatrix} g_1'' \\ \vdots \\ g_K'' \end{bmatrix}$$

where
$$\nabla h = \begin{bmatrix} \frac{\partial h}{\partial y_1} \\ \vdots \\ \frac{\partial h}{\partial y_k} \end{bmatrix} \qquad g' = \begin{bmatrix} g_1' \\ \vdots \\ g_k' \end{bmatrix} \qquad g'' = \begin{bmatrix} g_1'' \\ \vdots \\ g_k'' \end{bmatrix}$$

$$\nabla^2 h = \begin{bmatrix} \frac{\partial^2 h}{\partial y_i \partial y_j} \end{bmatrix}_{k \times k}$$
positive or negative semi-definite

Assume g_i is convex, $g_i''(x) \geq 0$

If $\nabla^2 h(\cdot) \geq 0$ then

$$g'(x)^T \nabla^2 h(g(x)) \, g'(x) \geq 0$$

If in addition $\nabla h(g(x)) \geq 0$ for each element, then

$$\nabla h(g(x))^T g''(x) \geq 0$$

We then obtain the following rule

- The convexity of f must follow that of h
- If h & g are of the same type, need h increasing in every argument
- If h & g are of different types, need h decreasing in every argument.

Finally, if $\text{dom } h \neq \mathbb{R}^k$, we will need to replace h' by extend-value function $\tilde{h}$.

Example:

ⓐ $h(z) = z_{[1]} + \cdots + z_{[r]}$, the sum of the r largest component. skip

- $h(z)$ is convex

$g_1(x), \cdots, g_k(x)$ are convex.

$\Rightarrow h(g_1(x), \cdots, g_k(x))$ is convex because $h(z)$ is non-decreasing in each argument.

ⓑ $h(z) = \log\left(\sum_{i=1}^{k} e^{z_i}\right)$

$g_1(x), \cdots g_k(x)$ are convex

$\Rightarrow h(g_1, \cdots, g_k) = \log\left(\sum_{i=1}^{k} e^{g_i(x)}\right)$ is convex.

ⓒ $f(x,u,v) = \sqrt{uv - x^T x}$ $\quad x \in R^n, \; u, v \in R$

on $u>0, v>0, uv > x^T x$

See Boyd Ex 3.22(b), p 117

$$f(x,u,v) = \sqrt{u\left(v - \frac{x^T x}{u}\right)}$$
$$= \sqrt{g_1 \cdot g_2}$$

$g_1(x,u,v) = u, \quad g_2(x,u,v) = v - \frac{x^T x}{u}$

- $\sqrt{g_1 g_2}$ is concave in g_1, g_2

- $g_2(x,u,v)$ is concave in (x,u,v)
- $g_1(u)=u$ is concave
- $\sqrt{g_1 g_2}$ extended to R^2 is non-decreasing in g_1 & g_2.

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Summary

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- Convex functions

$$f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$$

- First order condition

$$f(y) \ge f(x) + f'(x)(y-x)$$

Second order condition

$$f''(x) \ge 0$$

- Operations
    - Non-negative weighted sum
    - Affine change of variable
    - Pointwise maximum
    - Pointwise minimum (more demanding)
    - Composition
    - Perspective

Jensen's inequality - skip

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If a function f is convex, then

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$

for any $0 \le \theta \le 1$.

Easily extended to finite convex combinations

$$f(\theta_1 x_1 + \cdots + \theta_k x_k)$$
$$\le \theta_1 f(x_1) + \cdots + \theta_k f(x_k)$$

for all $\theta_1 + \cdots + \theta_k = 1$, $0 \le \theta_1, \ldots, \theta_k \le 1$

Can also be extended to infinite convex combinations

Let $\theta_1, \cdots, \theta_k, \cdots \ge 0$ satisfies

$$\sum_{k=1}^{+\infty} \theta_k = 1$$

Let $x_1, \cdots x_k, \cdots \in \operatorname{dom} f$, which is convex and the limit $\sum_{k=1}^{+\infty} \theta_k x_k$ exists

Then ① $\sum_{k=1}^{+\infty} \theta_k x_k \in \operatorname{dom} f$

② $f\left(\sum_{k=1}^{+\infty} \theta_k x_k\right) \le \sum_{k=1}^{+\infty} \theta_k f(x_k)$.

The proof of part 1 will need the Separation Theorem (to be discussed later)

The proof of part 2 uses the first-order condition (assuming f is differentiable), or use subgradients (to be discussed later).

We can also extend to integrals

- Suppose $p(\cdot)$ is a density on $S \subseteq \operatorname{dom} f$

  $p(x) \geq 0$ and $\int_S p(x)\, dx = 1$

- If f is convex, then

  $$f\left(\int_S p(x) \cdot x \, dx\right) \leq \int_S f(x)\, p(x)\, dx$$

Same for expectations

$$f(E X) \leq E f(X).$$

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Convex optimization problems

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- A convex optimization problem minimizes a convex function over a convex set

$$\min \; f_0(x) \quad \leftarrow \text{convex func.}$$
$$\text{sub to} \; x \in C \quad \leftarrow \text{convex set.}$$

- However, for the most part we will use the following "standard form":

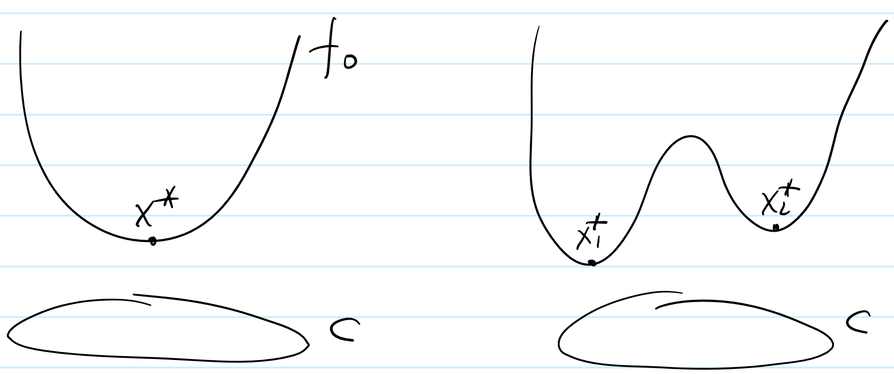
Standard form:

$$\begin{aligned} \min \quad & f_0(x) \\ \text{sub to} \quad & f_i(x) \le 0, \quad i = 1, \cdots, m \\ & h_i(x) = 0, \quad i = 1, \cdots, p \end{aligned}$$

where f_0, f_i are convex functions
h_i are linear functions

- x: variable to optimize
  $f_0(x)$: objective
  $f_i(x) \le 0$: inequality constraints
  $h_i(x) = 0$: equality constraints

- A key feature is that any local optimum must be a global optimum



- What about concave functions?
  - min a concave func won't work

  - max a concave func f_0 is the same as min the convex func $-f_0$
  - $f_1(x) \geq 0$ for a concave func is the same as $-f_1(x) \leq 0$ for the convex func. $-f_1(x)$.
- Why $h_i(x)$ must be linear?
- Some properties of convex opt. problems hold for the general form, but some required the stricter form.

- Domain of the problem:

$$D = \bigcap_{i=0}^{m} \mathrm{dom} f_i \bigcap_{i=1}^{p} \mathrm{dom}\, h_i$$

- A point x is feasible if $x \in D$ and x satisfies the constraints $f_i(x) \leq 0$ and $h_i(x) = 0$.

- Optimal value

$$p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0,\ i=1,\cdots,m,\ h_i(x) = 0,\ i=1,\cdots,p \}$$

we allow
- $p^* = +\infty$ (if no point x is feasible)
- $p^* = -\infty$ (if there exists a sequence of

feasible x_k, such that $f(x_k) \to -\infty$).

— Optimal point:

If the optimal value p^* is attained at feasible point x^*

$\Rightarrow$ x^* is an optimal point.

— We have taken the inequality constraints as "$\leq$". Although some results also apply for "$<$".

The "$\leq$" part often implies that the feasible set is closed. Hence, the optimal value is always attainable if $f_0(x)$ is continuous.

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Continuity of convex functions

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Skip. Related to directional derivative

The simple property that

$$f(\theta x + (1-\theta) y) \geq \theta f(x) + (1-\theta) f(y)$$

is in fact very strong and have important implications both geometrically & algebraically.

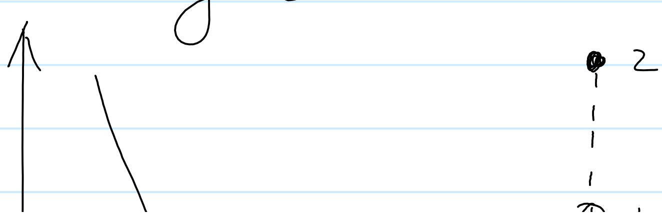
For example:

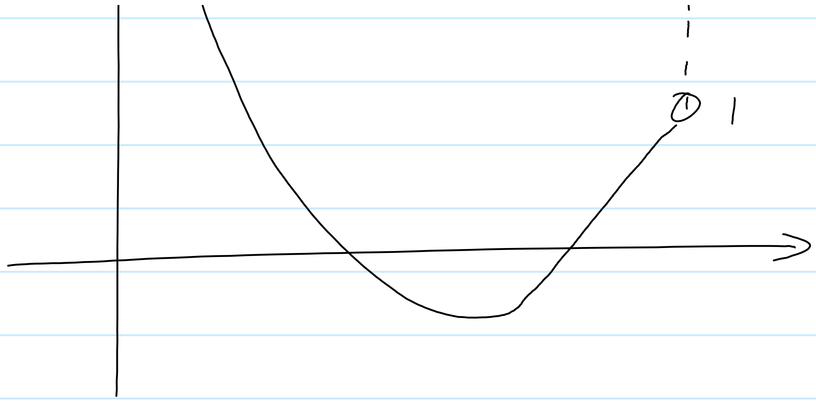
Continuity:

Let f be a convex function, then f is continuous on the interior of its domain

Discontinuity cannot occur in the interior of the domain.

Discontinuity can occur at the boundary.





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More on this when we discuss separation Theorems.