Lec7-mwf

Tuesday, January 25, 2011 11:03 PM

HW3 is assigned.

Project 1 is assigned. Can work on Problem 1(a).

We have studied ways to check convex functions:

- First-order condition
- Second-order condition

We have also several operations that preserve convexity of functions:

- Non-negative weighted sum
- Affine change of arguments
- Pointwise maximum

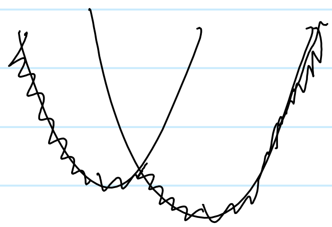
https://www.youtube.com/watch?v=aGZlP3jieRI

Minimizations

Wednesday, January 14, 2009 10:34 AM

④ Minimization of a jointly convex function over a convex set.

In general, minimization of convex function is NOT a convex function.



However, there is an important exception.

If f is convex in (x, y) jointly and C is a convex non-empty set, then

$$g(x) \triangleq \inf_{y \in C} f(x, y). \quad \text{is convex.}$$

Difference with maximization

- f must be jointly convex
- Must take the minimum over a convex set (e.g. can't just take two functions)

Proof: Assume that the infimum is always attained. (What to do if this assumption does not hold?)

Take $g(x_1) \triangleq \inf_{y \in C} f(x_1, y) = f(x_1, y_1)$

$g(x_2) \triangleq \inf_{y \in C} f(x_2, y) = f(x_2, y_2)$

We want to show that

$$g(\theta x_1 + (1-\theta)x_2) \le \theta g(x_1) + (1-\theta) g(x_2)$$

$$\triangle\|$$

$$\inf_{y \in C} f(\theta x_1 + (1-\theta)x_2, y) \le \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2)$$

It suffices to find a $y \in C$ such that

$$f(\theta x_1 + (1-\theta)x_2, y) \le \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2)$$

Taking $y = \theta y_1 + (1-\theta) y_2$, we have

$y \in C$ and the result follows from the convexity of f.

Examples:

ⓐ Distance to a set

$$\text{dist}(x, C) \triangleq \inf_{y \in C} \|x - y\|$$

If C is a convex set, then $\text{dist}(x, C)$ is convex.

(5) $\displaystyle g(x) = \inf f_0(y)$

sub $f_1(y) \le x$

If f_0 & f_1 convex, then g is convex

Proof:

Define $f(x,y) = \begin{cases} f_0(y) & \text{if } f_1(y) \le x \\ +\infty & \text{otherwise} \end{cases}$

- This is the "extended" function that we will see again in composition

Then $f(x,y)$ is convex in (x,y) (proved by yourself)

& $\displaystyle g(x) = \inf_{y \in R} f(x,y)$

- Useful when we study strong duality.

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Perspective of a function

Saturday, January 17, 2009 2:52 PM

If f is convex, then its perspective

$$g(x,t) = t f\left(\frac{x}{t}\right), \quad \operatorname{dom} g = \left\{\frac{x}{t} \in \operatorname{dom} f,\ t > 0\right\}$$

is also convex.

Proof: Recall perspective mapping

$$(x,t) \to \frac{x}{t}$$

$$\operatorname{epi} g = \left\{(x,t,s) \,\middle|\, s \geq t f\left(\frac{x}{t}\right)\right\}$$

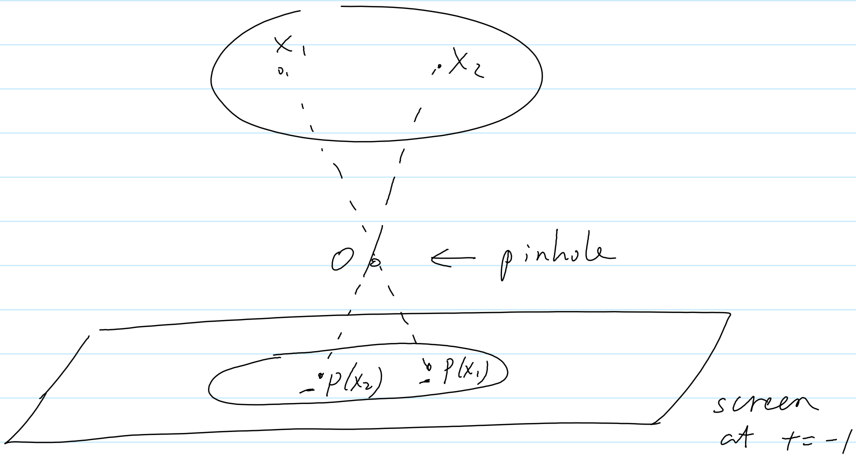
$$= \left\{(x,t,s) \,\middle|\, \frac{s}{t} \geq f\left(\frac{x}{t}\right)\right\}$$

$$\operatorname{epi} f = \left\{(x, s) \,\middle|\, s \geq f(x)\right\}$$

Hence, epi g is the inverse perspective-mapping of epi f.

$\Rightarrow$ epi g is convex.

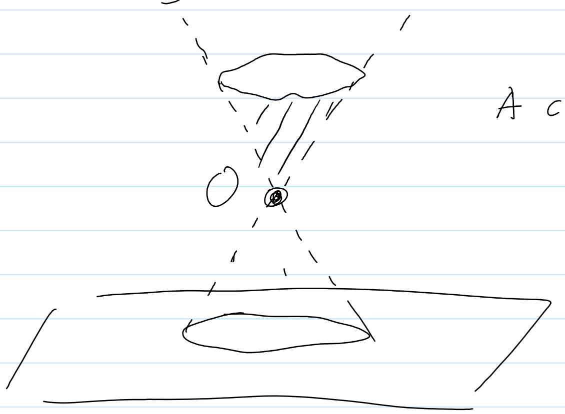
— We discuss earlier that the perspective mapping preserves convexity.



— It turns out that the inverse of

- It turns out that the inverse of a perspective mapping also preserves convexity

See Boyd p40.



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More directly:

Let $s_1 = g(x_1, t_1) = t_1 f\left(\frac{x_1}{t_1}\right) \Rightarrow \frac{s_1}{t_1} = f\left(\frac{x_1}{t_1}\right)$

$s_2 = g(x_2, t_2) = t_2 f\left(\frac{x_2}{t_2}\right) \Rightarrow \frac{s_2}{t_2} = f\left(\frac{x_2}{t_2}\right)$

Consider

$$\theta s_1 + (1-\theta) s_2, \quad \theta x_1 + (1-\theta) x_2, \quad \theta t_1 + (1-\theta) t_2$$

We want to show that

$$\theta s_1 + (1-\theta) s_2 \geq [\theta t_1 + (1-\theta) t_2] f\left(\frac{\theta x_1 + (1-\theta) x_2}{\theta t_1 + (1-\theta) t_2}\right)$$

$$\Leftrightarrow \frac{\theta s_1 + (1-\theta) s_2}{\theta t_1 + (1-\theta) t_2} \geq f\left(\frac{\theta x_1 + (1-\theta) x_2}{\theta t_2 + (1-\theta) t_2}\right)$$

We hope to write $\frac{\theta u_1 + (1-\theta) u_2}{\theta t_1 + (1-\theta) t_2}$ as a convex combination of $\frac{u_1}{t_1}$ & $\frac{u_2}{t_2}$ (u_1/u_2 can be either s_1/s_2 or x_1/x_2).

If we can do that, then we will just need

$$\gamma \frac{s_1}{t_1} + (1-\gamma)\frac{s_2}{t_2} \geq f\left(\gamma \frac{x_1}{t_1} + (1-\gamma)\frac{x_2}{t_2}\right)$$

which follows from the convexity of f.

This convex combination is

$$\frac{\theta u_1 + (1-\theta)\, u_2}{\theta t_1 + (1-\theta) t_2} = \frac{u_1}{t_1} \underbrace{\frac{\theta t_1}{\theta t_1 + (1-\theta) t_2}}_{\gamma} + \frac{u_2}{t_2} \underbrace{\frac{(1-\theta) t_2}{\theta t_1 + (1-\theta) t_2}}_{1-\gamma}$$

so we are done.

γ is independent of u.

Example:

(a) $R(P) = \log\left(1 + \frac{P}{N}\right)$

- serve the user for only $t<1$ fraction of time at average power P
- Peak power is P/t
- Avg rate $= R(P, t) = t \cdot \log\left(1 + P/t\right)$
  - jointly concave in P, t.

(2) $\frac{x^2}{t}$, $\frac{x^4}{t^3}$, $t>0$, are both convex

$\frac{x^T x}{v}$ is convex

$\sqrt{xy}$ is concave

(3) KL - divergence $\quad \sum_i p_i \log \frac{p_i}{q_i}$

Combined with an affine mapping,

If f is convex, then

$$g(x) = (c^T x + d) f\left(\frac{Ax+b}{c^T x + d}\right) \text{ is convex}$$

with $\operatorname{dom} g = \left\{ c^T x + d > 0,\ \frac{Ax+b}{c^T x+d} \in \operatorname{dom} f \right\}$

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Compositions

Wednesday, January 14, 2009 10:48 AM

Consider $f(x) = h(g(x))$

We know that

- if h is non-negative weighted sum, and g is convex $\Rightarrow$ f is convex
- if h is convex and g is affine $\Rightarrow$ f is convex

What if more general cases?

First, assume h, g are both $R \to R$

and they are both differentiable.

- Further, let us study first the case $\operatorname{dom} h = R$.

Using the chain-rules

$$f'(x) = h'(g(x)) \cdot g'(x)$$

$$f''(x) = h''(g(x)) \cdot \underbrace{(g'(x))^2}_{\geq 0} + h'(g(x))\, g''(x)$$

Assume g is convex, $g''(x) \geq 0$

If $\underset{h'' \geq 0}{h \text{ is convex}}$, $\underset{h' \geq 0}{h \text{ is non-decreasing}}$ $\Rightarrow f$ is convex

$\underset{h'' \leq 0}{h \text{ is concave}}$, $\underset{h' \leq 0}{h \text{ is non-increasing}}$ $\Rightarrow f$ is concave

Assume g is concave, $g''(x) \leq 0$

If h is concave, h is non-decreasing $\Rightarrow f$ is concave

h is convex, h is non-increasing $\Rightarrow f$ is convex

Note:

- The convexity of f must follow that of h
- If h & g are of the same type, need h increasing (think positive weighted sum)
- If h & g are of different types, need h decreasing. (think negative weighted sum)

Examples:

(a) If g is convex, then $e^{g(x)}$ is convex.

(b) If g is convex, then $g^2(x)$ may not be convex

(c) If g is convex, $g \geq 0$, then $g^2(x)$ is convex

Hint: use $h(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$

If dom $h \neq R$, then it can be a bit trickier

- especially if the dom h has "artificial" restrictions
- Let us look at a counter example first.

Ex) $g(x) = x^2$

$h(x) = 0$ with dom $h = [1, 2]$

It seems that $f(x) = h(g(x))$ should be convex
- h is convex
- g is convex
- h non-decreasing.

However, $\mathrm{dom} f = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$

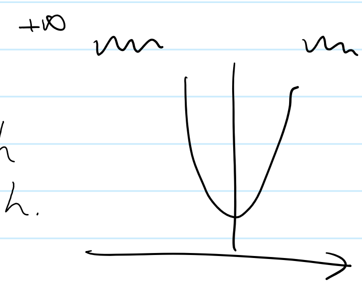
$\Rightarrow f$ cannot be convex.

"Safest" Approach:

If $\mathrm{dom}\, h$ is not $\mathbb{R}$, then we need to first extend h to $\tilde{h}$, the extended value function

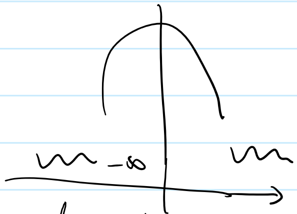
- We want $\tilde{h}$ to have the same convexity/concavity as h.

If h is convex



$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in \mathrm{dom}\, h \\ +\infty & \text{if } x \notin \mathrm{dom}\, h. \end{cases}$$

If h is concave



$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in \mathrm{dom}\, h \\ -\infty & \text{if } x \notin \mathrm{dom}\, h \end{cases}$$

Rule: If $\mathrm{dom}\, h$ is not $\mathbb{R}$, then in order to check convexity of the composed function $h(g(x))$, we need to use the properties of $\tilde{h}$.

Note that
- $\tilde{h}$ always preserve the convexity/concavity of h.
- If h is non-decreasing, $\tilde{h}$ is NOT necessarily

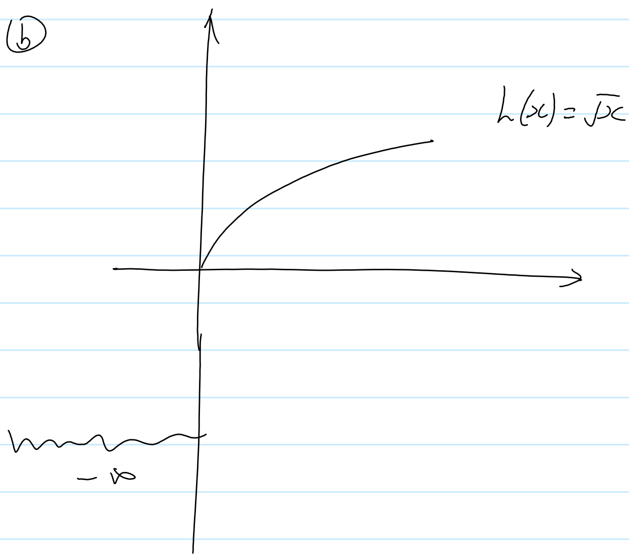
non-decreasing

(a)

$\hat{h}$ is NOT monotone

$\hat{h}$ does not make too much of the differences if the original domain is "natural"

(b)



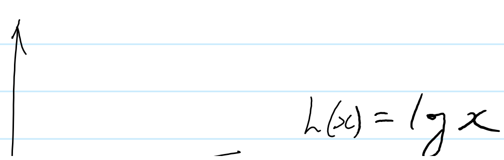
$\hat{h}$ is non-decreasing

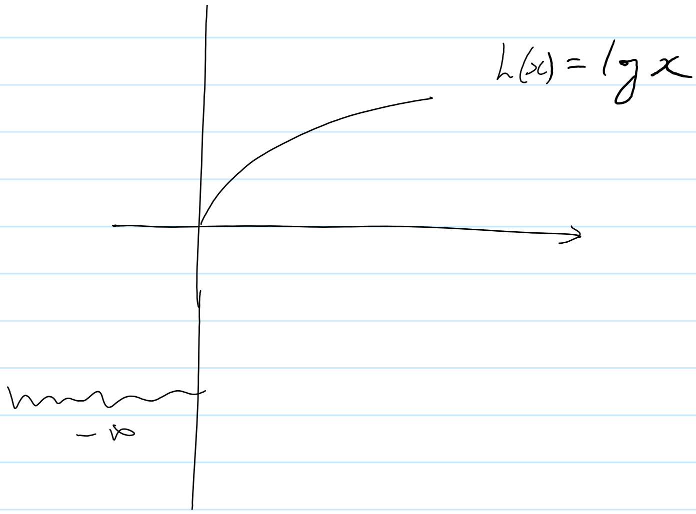
(c) $h(x) = \log x$: $\hat{h}$ is concave & non-decreasing

Examples:

① If $f(x)$ is concave, is $\log f(x)$ concave on $\{x \mid f(x) > 0\}$?

- $h(x) = \log x$ concave, but only on $x > 0$!!
- $\hat{h}(x)$ is concave & non-decreasing.



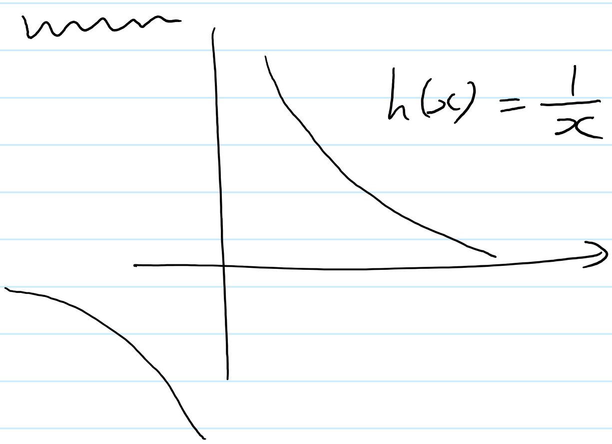


$\hat{h}$ is non-decreasing

$\Rightarrow$ $\log g(x)$ is concave on $\underbrace{\{x \mid g(x) > 0\}}_{\text{convex set}}$

② If $g(x)$ is concave, is $\frac{1}{g(x)}$ convex? No

Is $\frac{1}{g(x)}$ convex on $\{x \mid g(x) > 0\}$? Yes.



③ $h(x) = x^{3/2}$ on $[0, +\infty]$, is $h(x^2 - 4x)$ convex?

– $\hat{h}$ is not non-decreasing $\Rightarrow$ No

– Domain is $x \in [-\infty, 0) \cup (4, +\infty)] \Rightarrow$ Non-convex

④ $h(x) = \begin{cases} x^{3/2} & x \geq 0, \end{cases}$

| 0   otherwise

Is $h(x^2 - 4x)$ convex?
Yes. h is convex & non decreasing (no extension needed).

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