Lec4

Tuesday, January 18, 2011 7:58 PM

https://www.youtube.com/watch?v=wQylqaCl8Zo&t=178s

Summary

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Convex sets

$$\theta x_1 + (1-\theta) x_2 \in C \qquad 0 \le \theta \le 1$$

Key examples

- Affine sets, subspaces
- Balls, ellipsoids
- hyperplanes, half-spaces
- polyhedra

Operations that preserve convexity

- Intersections
- Affine mappings
- Perspectives

Convex functions

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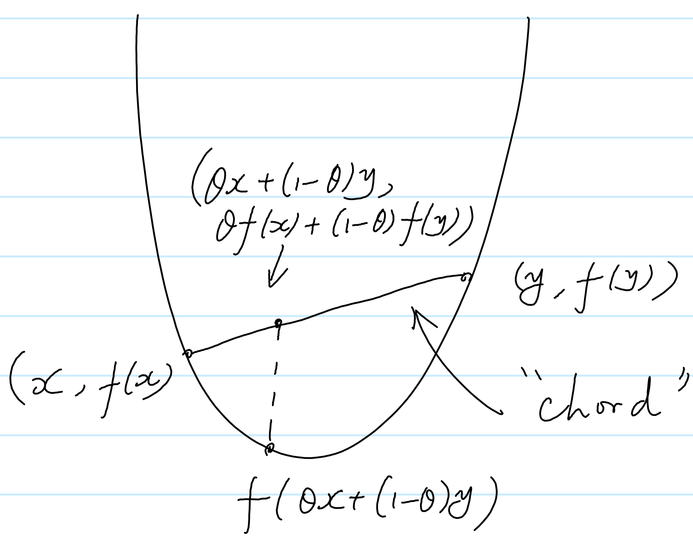
A function $f : R^n \to R$ is convex if its domain $\mathrm{dom}\, f$ is a convex set and

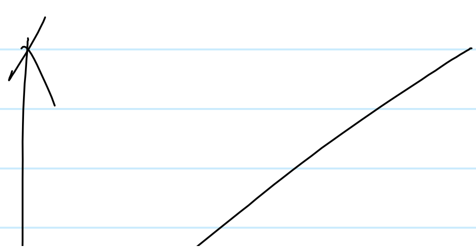
$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y) \quad (*)$$

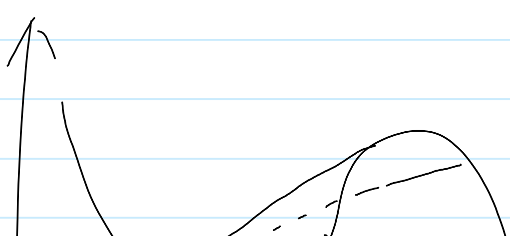
for all $x, y \in \mathrm{dom}\, f$, $0 \leq \theta \leq 1$.

Geometrically, this means that the line segment between $(x, f(x))$ & $(y, f(y))$, which is the chord from x to y, lies above the graph of f.

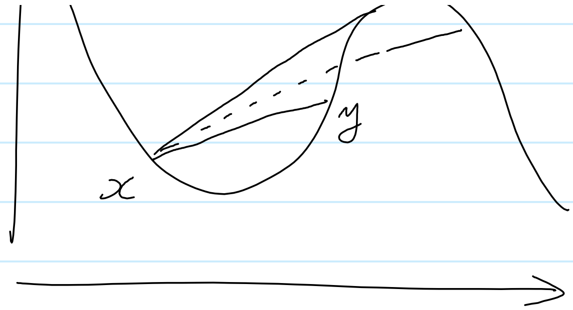


Note that by this definition, an affine function is also convex.

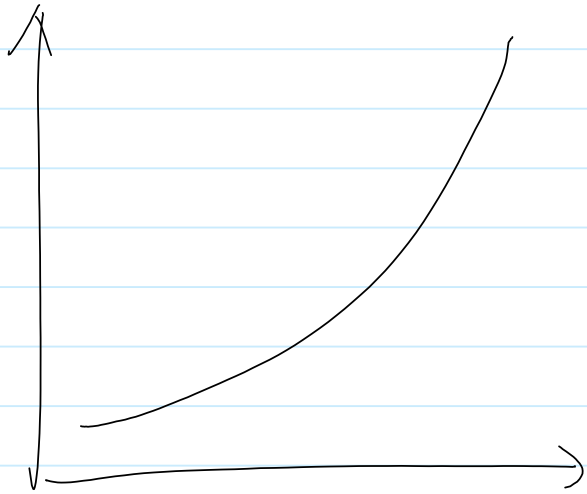




Convex & Affine



Not Convex



Convex & Increasing

Convex & decreasing

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f is concave if $-f$ is convex

con**V**ex

con**cave**

Affine functions are both convex & concave.

A function is strictly convex if strict inequality holds in (*) whenever $x \neq y$ & $0 < \theta < 1$.

- does not contain any affine parts.

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Relationships

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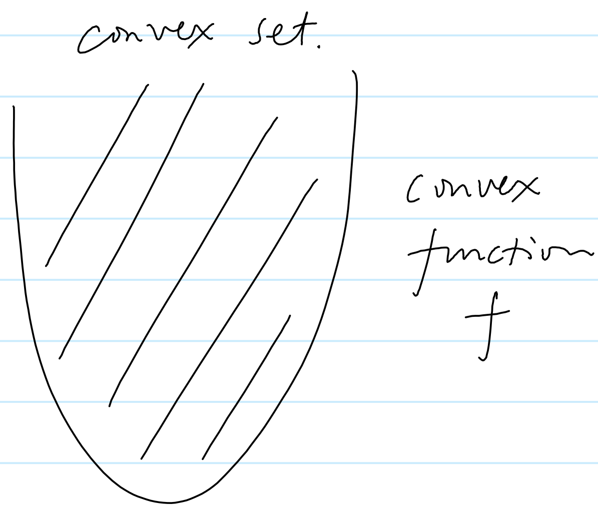
<u>Relationship between convex functions & convex sets.</u>

① Define the epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$ as

$$\text{epi } f = \{ (x, t) \mid x \in \text{dom } f,\ t \geq f(x) \}$$

A function is convex if and only if its epigraph is a convex set.



Proof: By definition

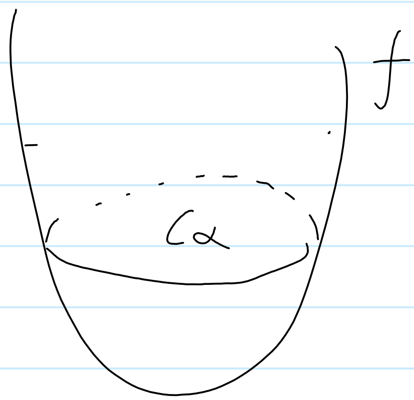
- This will be useful in proving strong duality

Ⓠ Is the epigraph of a concave function a convex set?

(2) Define the sub-level set

$$C_\alpha = \{x \in \text{dom} f \mid f(x) \le \alpha\}$$

If f is a convex function, then C_α is a convex set for any α



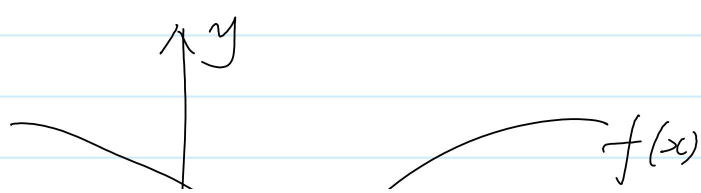
Proof: By definition.

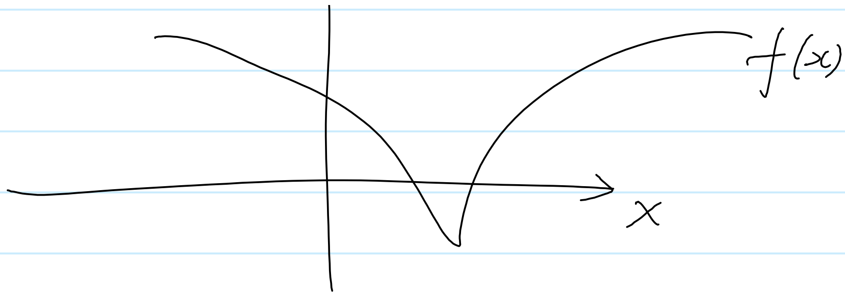
- This is actually the main convex sets that we will encounter later.

(Q) Is the sublevel set of a concave function a convex set?
- $\{x \mid f(x) \ge \alpha\}$

(Q) Is the converse true? In other words, if the sublevel set C_α of a function f is convex for any α, then is the function f convex?

(A) No.





(Exercise)

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Restriction to a line

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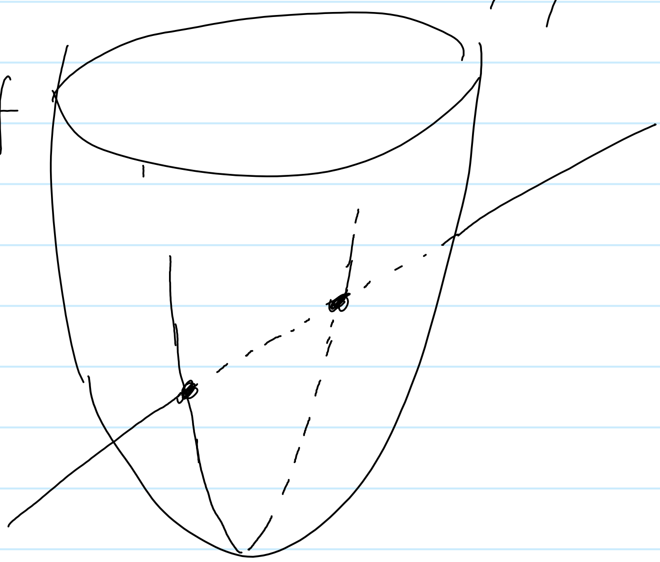
A function is convex if and only if it is convex when restricted to any line that intersects its domain

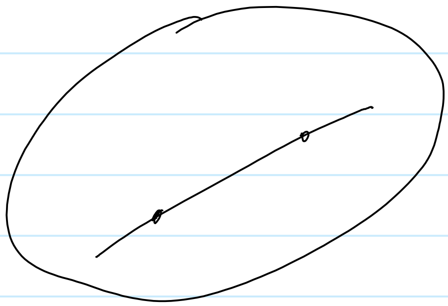
f is convex $\Leftrightarrow$ for all $x \in \mathrm{dom} f$ and all v

$g(t) = f(x + tv)$ is convex on $\{t \mid x + tv \in \mathrm{dom} f\}$

f



dom f



This is not surprising because, to check convexity, we only need to check on straight lines.

A function is convex if, for any two points, the chord is above the function. But the chord is also above that part of the function restricted to the same line.

This is useful because we can reduce the problem of checking convexity of any function to that of checking convexity of functions of one variable.

(Q) A function $f(x,y)$ is convex w.r.t. each variable (x or y), is f a convex function of x <u>and</u> y?

(A) No.

- Example $f(x,y) = xy$.

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- product of convex functions may not be convex.

Conditions for convexity

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Often it is not easy to verify convexity of a function using the definition directly.

The following conditions will be handy.

- First-order conditions ∇f
- Second-order conditions $\nabla^2 f$ } calculus
- Composition that preserves convexity.

① First-order condition.

If f is differentiable in its domain $\mathrm{dom} f$, then f is convex if and only if $\mathrm{dom} f$ is a convex set and

$$f(y) \geq f(x) + [\nabla f(x)]^T (y-x) \qquad (*)$$

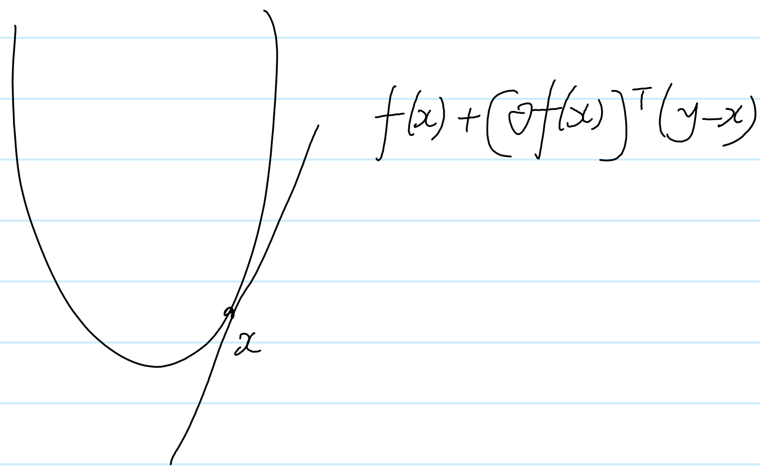
for all $x, y \in \mathrm{dom} f$.

f is strictly convex if and only if $\mathrm{dom} f$ is convex and

$$f(y) > f(x) + [\nabla f(x)]^T (y-x)$$

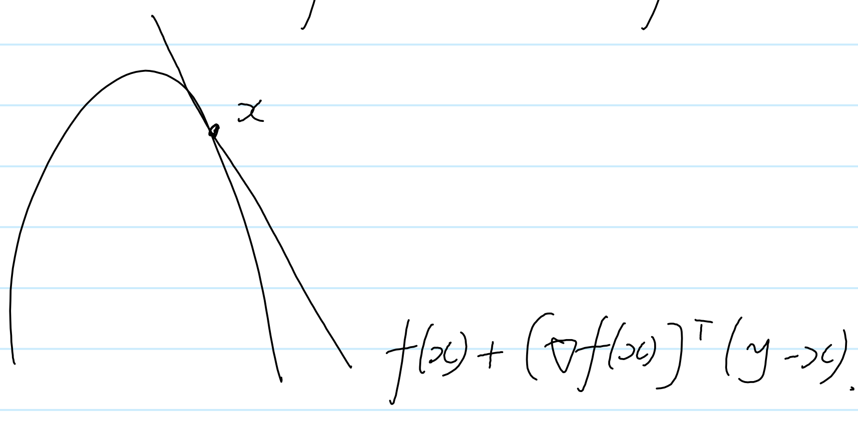
for all $y \neq x$, $x, y \in \mathrm{dom} f$.

Note that $f(x) + [\nabla f(x)]^T (y-x)$ defines a hyperplane (of the variable y) tangent to $f(x)$ at $(x, f(x))$.



Hence, the condition states that the rest of the function must be above the hyperplane.
— "supporting hyperplane".

A similar result holds for concave functions.



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Proof of first-order condition

Monday, January 12, 2009 2:42 PM

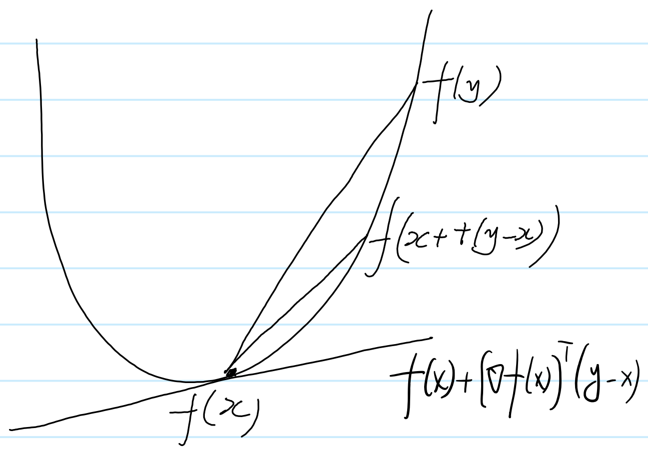
Sketch: Let us focus on the one-dimensional case $f: R \to R$.

"$\Rightarrow$" direction: If f is convex, then (*) holds.

To see this, let

$$g(t) = \frac{f(x + t(y-x)) - f(x)}{t \cdot (y-x)}, \quad 0 < t \le 1.$$

Note that $g(t)$ can be viewed as an approximation of the derivative $f'(x)$.



If $t=1$, the ratio is

$$g(1) = \frac{f(y) - f(x)}{y-x}$$

If $t \to 0$,

$$g(t) \to f'(x).$$

Graphically, it is intuitive that, as $t \downarrow 0$

$$g(t) \downarrow f'(x)$$

— can be shown through convexity

For us, in order to show that

$$\frac{f(y) - f(x)}{y-x} \ge f'(x),$$

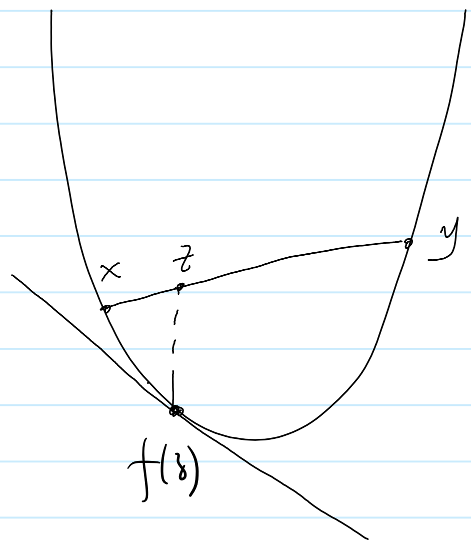
it suffices to show that $g(t) \le g(1)$ for all t

i.e. $$\frac{f(x + t(y-x)) - f(x)}{t(y-x)} \le \frac{f(y) - f(x)}{y-x}$$

(homework).

"$\Leftarrow$" direction: If $(*)$ holds, f is convex.

For any x, y, let $z = \theta x + (1-\theta) y$



(homework)

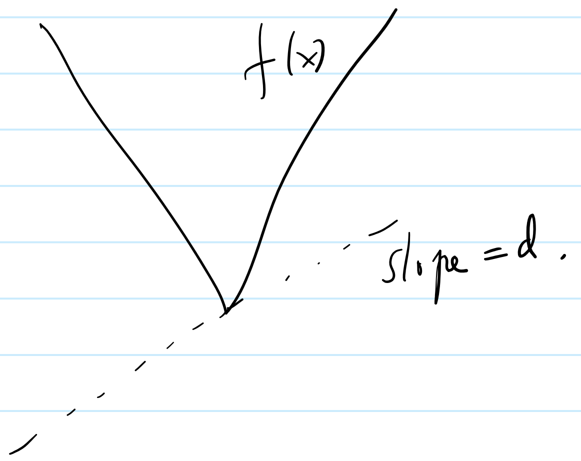
Q What if the function is not differentiable?

- From the proof in the "$\Leftarrow$" direction, the only property that we need is that there exists some vector d for every x

$$f(y) \geq f(x) + d^T (y-x) \quad \text{for all } y.$$

↑ exists even if f is not differentiable at x

$\Rightarrow$ subgradients



- first-order condition is useful when the second-order derivative is not easy to derive
  - e.g. in some of the proofs for duality.