Lec3

Wednesday, January 14, 2009 10:29 PM

HW1 is on the web. Due in 1 week.

Important examples of convex sets

Friday, January 09, 2009 6:29 PM

In convex optimization, the constraint set must be convex.

It would be helpful to identify convex sets quickly.

① ϕ, R^n

② Any affine set / Subspace

	$\theta_1 x_1 + \cdots + \theta_k x_k \in C$ for	
Subspace	Any $\theta_1, \cdots \theta_k$	More restrictive
Affine set	Any $\theta_1 + \cdots + \theta_k = 1$	less restrictive
Convex set	Any $\theta_1 + \cdots + \theta_k = 1$, $0 \le \theta_1, \cdots \theta_k \le 1$	Least restrictive

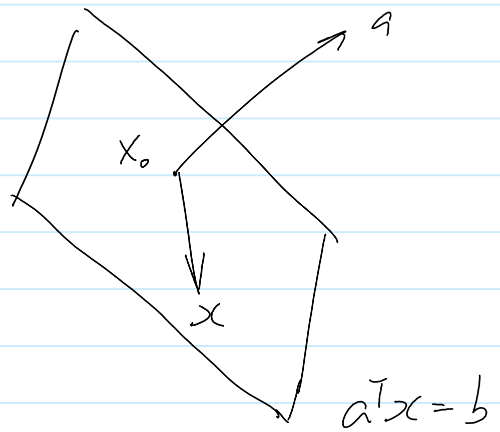
③ Lines, rays, line segments

④ Hyper-planes

$$\{x \mid a^T x = b\}$$

Half-space

$$\{x \mid a^T x \le b\} \qquad \{x \mid a^T x < b\}$$



$a(x - x_0) = 0$

$\Rightarrow (x - x_0)$ is orthogonal to $\vec{a}$

(5) Balls

$$\|x - x_0\|_2 \le r$$

- can replace by any "norm" that satisfies the triangular-inequality

Ellipsoid.

$$(x - x_0)^T P^{-1} (x - x_0) \le 1$$

where P is symmetric and positive semi-definite

- A matrix is positive semidefinite if

$$x^T A x \ge 0 \quad \text{for all } x$$

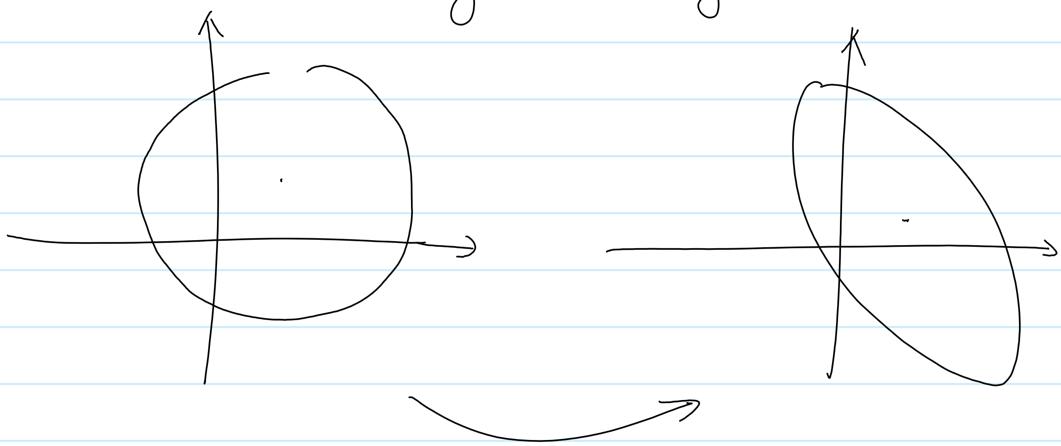
e.g. $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Any semi-definite matrix A can be written as

$$A = V \Lambda V^{-1}$$

where Λ is an diagonal matrix with non-negative diagonal elements

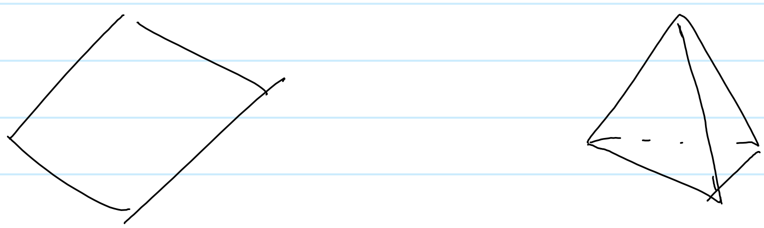


will see later that convexity is preserved through linear transformation.

⑥ Polyhedra: the intersection of a finite # of halfspaces & hyperplanes

$$\{x \mid a_j^T \le b_j, \ j = 1, 2, \cdots, J$$
$$c_j^T x = d_j, \ j = 1, 2, \cdots, k\}$$

$$= \{x \mid Ax \le b, \ Cx \le d\}$$



will see soon that convexity is preserved under intersection.

The converse is also true: Any convex set is the intersection of hyperplanes & half-spaces (could be infinitely many).

- In the forum though, most convex sets are described by constraint equations
  - $h(x) = 0$ for some linear function
  - $f(x) \leq 0$, < 0 for some convex function.

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Operations

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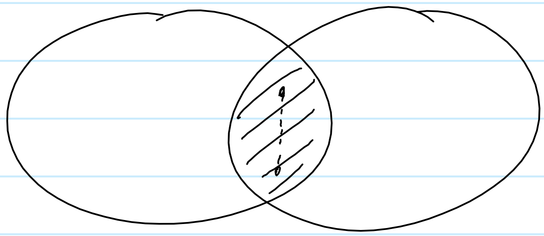
Operations that preserve convexity

① Intersection.

any index set

If C_i is a convex set for all $i \in I$,
then $\bigcap_{i \in I} C_i$ is convex.

<u>Intuition</u>



Useful when we laid down many constraints
that need to be satisfied simultaneously.

Note: there could be <u>uncountably</u> many C_i's.

- Comparable to "any intersection of closed sets is a closed set".
- Again, "convexity" seems to replace "closeness"

Very useful when we keep adding constraints to an optimization problem!

- Just make sure each one is convex!

Examples:

ⓐ Polyhedra.

(b) (Infinite # of intersections)

The set of positive semi-definite matrices is convex.

A symmetric $n \times n$ matrix $\underline{X}$ is positive semi-definite if

$$v^T X v \geq 0 \quad \text{for all vector } v$$

It is positive-definite if

$$v^T X v > 0 \quad \text{for all } v \neq 0$$

Proof: The set of positive semi-definite matrices can be written as

$$\bigcap_{v \in \mathbb{R}^n} \{ \underline{X} \mid \underline{X} \text{ symmetric } \& \ v^T \underline{X} v \geq 0 \}$$

For each v, since $v^T \underline{X} v$ is a linear function of $\underline{X}$, hence the set

$$\{ \underline{X} \mid \underline{X} \text{ symmetric } \& \ v^T \underline{X} v \geq 0 \}$$

is a halfspace.

$$\Rightarrow \bigcap_{v \in \mathbb{R}^n} \{ \underline{X} \mid \underline{X} \text{ symmetric } \& \ v^T \underline{X} v \geq 0 \}$$

is convex.

Alternate proof: by definition.

③ The set

$$S = \left\{ x \in R^m \;\middle|\; |P_x(t)| \le 1 \text{ for all } |t| \le \frac{\pi}{3} \right\}$$

where $P_x(t) = \sum_{k=1}^{m} x_k \cos kt$ ← trigonometric polynomials (e.g. in Fourier transform)

is convex.

Proof: For given t

$$\{x \mid |P_x(t)| \le 1\}$$

$$= \left\{ x \;\middle|\; -1 \le \sum_{k=1}^{m} x_k \cos kt \le 1 \right\}$$

is the intersection of two half-spaces

$\Rightarrow S$ is convex

Ⓠ Is the union of convex sets a convex set?

Ⓐ Usually not.

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Affine mapping

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A function $f: R^n \to R^m$ is affine if it is of the form

$$f(x) = Ax + b, \qquad A \in R^{m \times n}, \ b \in R^m$$

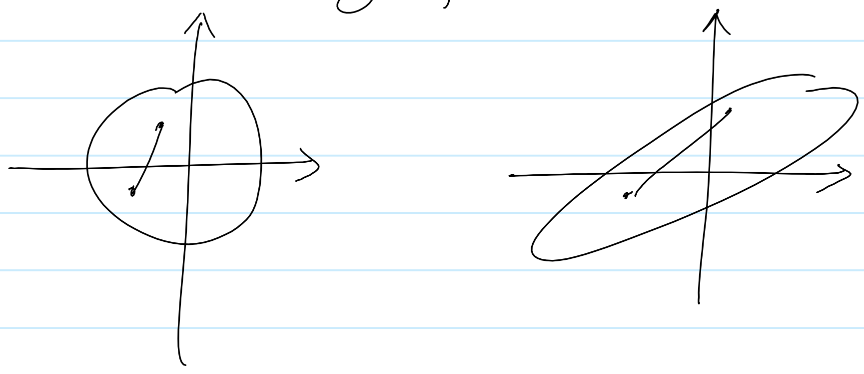
If $S \subseteq R^n$ is convex, and $f: R^n \to R^m$ is affine, then the image of S under f

$$f(S) = \{ f(x) \mid x \in S \}$$

is convex.

Intuition:

- Straight lines map to straight lines
- If x_1 can reach x_2 via straight lines w/o leaving S then $f(x_1)$ must reach $f(x_2)$ via straight lines w/o leaving $f(S)$.



Examples

(a) Balls & ellipsoid

- A ball is

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\}$$

↑ Euclidean norm

- An ellipsoid (more restrictive version)

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

where P is a positive definite matrix.

- Then $\mathcal{E}$ is the image of the unit ball $B(0,1)$ under the mapping

$$f(u) = P^{1/2} u + x_c$$

- Why?

$$(f(u) - x_c)^T P^{-1} (f(u) - x_c) = u^T P^{1/2} P^{-1} P^{1/2} u = u^T u \le 1$$

if $u \in B(0,1)$

$\Rightarrow \mathcal{E}$ is convex

Perspective mapping of a set

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The third kind of operations that preserve convexity is the perspective function & linear-fractionals.

Write $x = (z, t)$, where $z \in \mathbb{R}^n$, and $t \in \mathbb{R}^{++}$

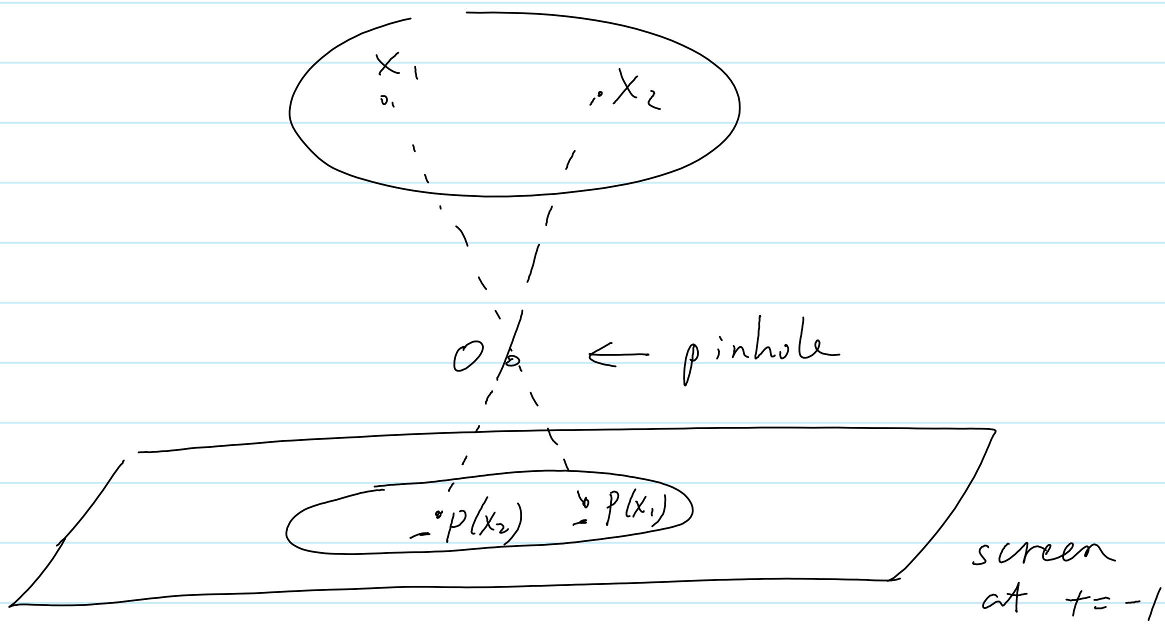
positive real-numbers

Let $p(x) \triangleq P(z,t) = z/t \in \mathbb{R}^n$.

Then $p(\cdot)$ is called a perspective function.

<u>Intuition</u>:

- scale the vector such that the last component is 1
- then drop the last component
- Can be interpreted as the action of a pin-hole camera



If $C \subset \mathbb{R}^n \times (\mathbb{R}^+ \setminus \{0\})$ is convex,
then $p(C) = \{ p(x) \mid x \in C \}$ is also
convex.

Proof: see Boyd P40. Again, straight
lines are mapped straight lines.

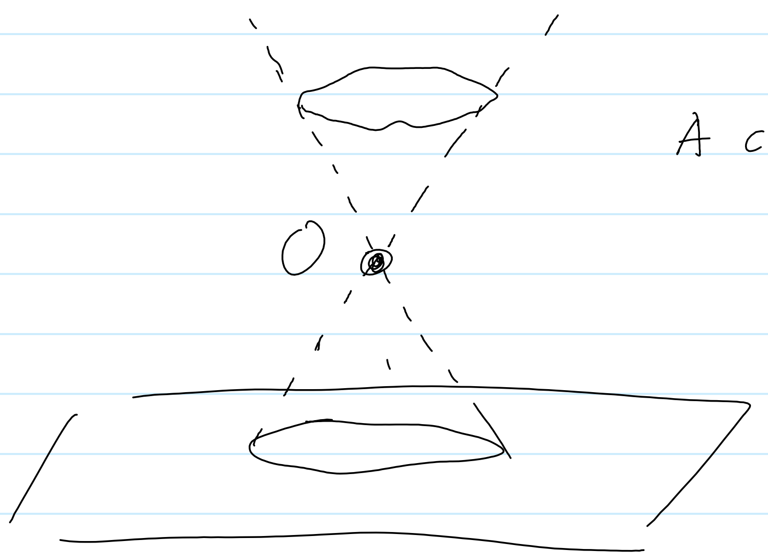
The Inverse is also true

If C is convex, define

$$p^{-1}(C) = \{ (x, t) \in \mathbb{R}^{n+1} \mid \tfrac{x}{t} \in C,\ t > 0 \}$$

then $p^{-1}(C)$ is convex.

Proof: See Boyd P40.



A cone

Linear-functional

$f(x) = Ax + b \qquad A \in \mathbb{R}^{m \times n},\ x \in \mathbb{R}^n,\ b \in \mathbb{R}^m$

$$f(x) = \frac{Ax+b}{c^T x + d} \qquad A \in R^{m \times n},\ x \in R^n,\ b \in R^m$$
$$c \in R^n,\ d \in R$$

If $X \subseteq \{x \mid c^T x + d \neq 0\}$ is convex

then $f(X)$ is convex

Proof. Write $f(x) = P(g(x))$

with $g(x) = (Ax+b,\ c^T x + d)$ - affine

& $P(u)$ is the perspective mapping.

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Summary

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Convex sets

$$\theta x_1 + (1-\theta) x_2 \in C \qquad 0 \leq \theta \leq 1$$

Key examples

- Affine sets, subspaces
- Balls, ellipsoids
- hyperplanes, half-spaces
- polyhedra

Operations that preserve convexity

- Intersections
- Affine mappings
- Perspectives