Lec20-mwf

Saturday, February 07, 2009 10:26 PM

Bring hand-out for separation

Separation theorem and duality

Wednesday, February 04, 2009 10:46 AM

- To show strong duality, we need to show that there exists $\lambda^* \geq 0$, ν^* such that

$$g(\lambda^*, \nu^*) = p^*.$$

- Then combined with weak-duality, we are done.

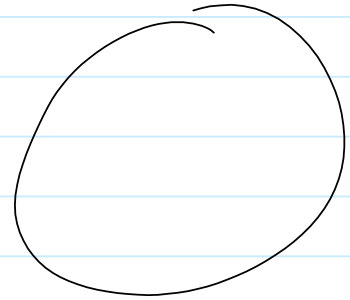
- To find these λ^*, ν^*, we need an important result in convex analysis, i.e., the separation theorem.

This material is technical, but in my opinion separation theorem plays an essential role in appreciating the implication of convexity.

As we briefly mentioned earlier, although convexity is defined algebraically, it is in fact a very strong condition that implies certain geometric / topological properties.

$x_1, x_2 \in C$

$\Rightarrow \theta x_1 + (1-\theta) x_2 \in C$



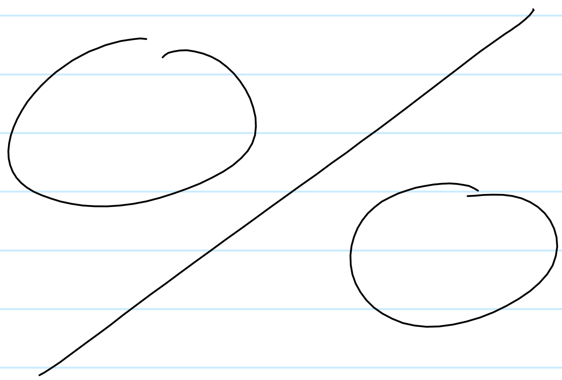
- Separation theorem is one such property.
    - Not only useful for duality
    - But also for subgradients and convex combination of infinite points.
- Continuity is another one.

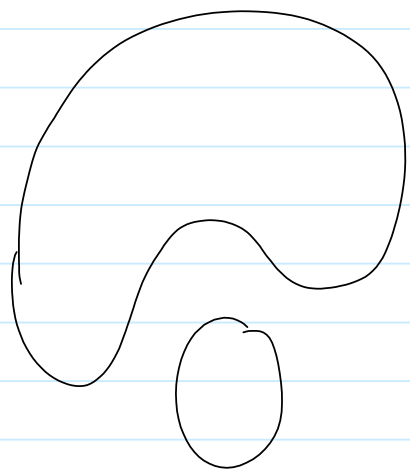
Separation theorem: intro

Wednesday, February 04, 2009 10:52 AM

- Rockafeller P95-101 (Use handout separation.pdf)

- Roughly speaking, if we have two convex sets that are disjoint from each other, then we can find a hyperplane that separates them.





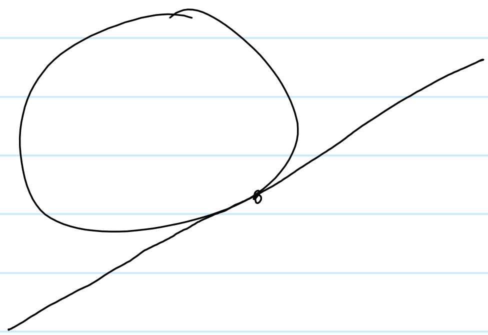
Not true if the sets are not convex!

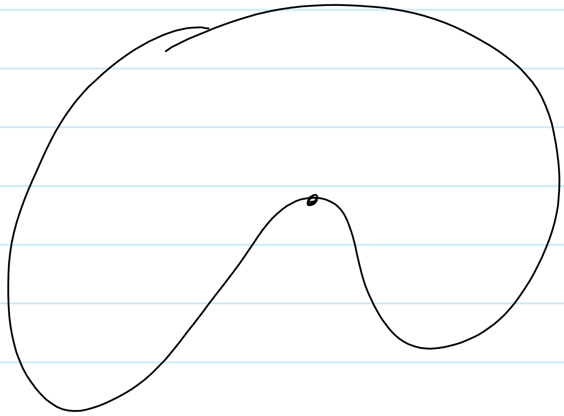
- A related result is on the supporting hyperplane

Supporting hyperplane:

- A supporting hyperplane of C is a hyperplane that contains a point in C and the closed half-space on one side of the hyperplane contains the set C.

- Roughly speaking, for every point on the boundary of a convex set C, we can find a supporting hyperplane that goes through this point.





Not true if the set is NOT convex!

Q Why are separation theorems important?

- They are "existence" theorems.
- Useful when we need to show that there exists a quantity that satisfies certain property

e.g. - strong duality
- subgradients.

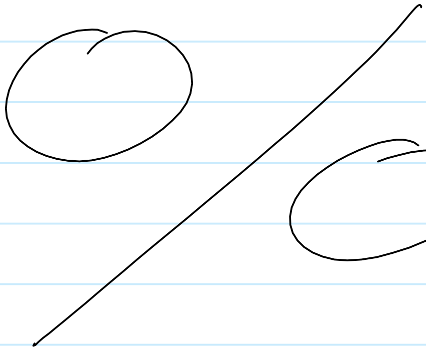
Our plan:
- state the theorems
- applications
- prove a weaker case. - skip

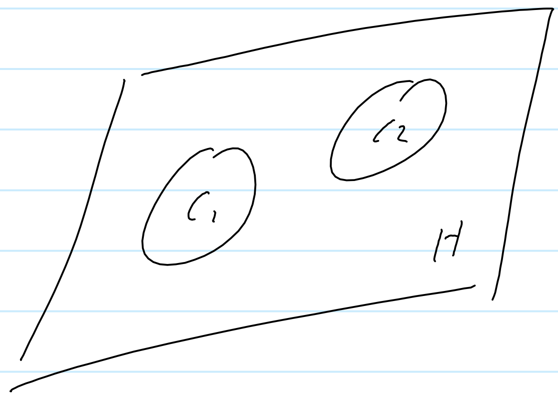
Definition of Separation

Thursday, February 05, 2009 9:28 AM

(Basic) Separation

- Let C_1 & C_2 be non-empty sets in R^n.
- A hyperplane is said to seperate C_1 & C_2 if C_1 is contained in one of the closed half-spaces associated with H, and C_2 is contained in the opposite closed half-space.
- Recall that a hyperplane is $\{bx = a\}$
    - $C_1 \subset \{x \mid bx \geq a\}$ & $C_2 \subset \{x \mid bx \leq a\}$
- While this is the most straight-forward definition, it can be too weak





This hyperplane also

separates C_1 & C_2!

Proper Separation

— It is said to separate C_1 & C_2 ~~properly~~, if C_1 & C_2 are not both actually contained in H itself.

— Recall that a hyperplane is $\{bx = a\}$

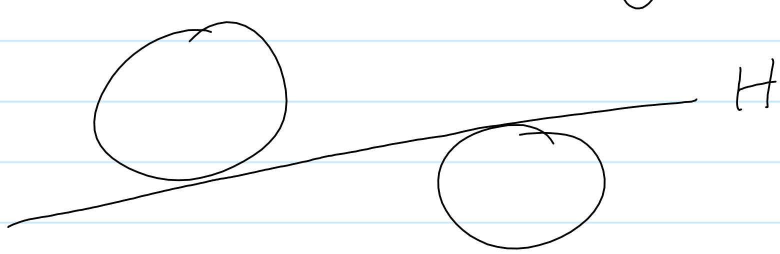
proper separation

$\Leftrightarrow$ there exists a vector b such that

① $\inf\{bx \mid x \in C_1\} \geq \sup\{bx \mid x \in C_2\}$

② $\sup\{bx \mid x \in C_1\} > \inf\{bx \mid x \in C_2\}$

— Note that the two sets may still touch H



even when they do not touch each other.

— Further, one of the two sets (not ~~both~~) may still be in H.

$\Rightarrow$ sometimes we want an even stronger notion of separation

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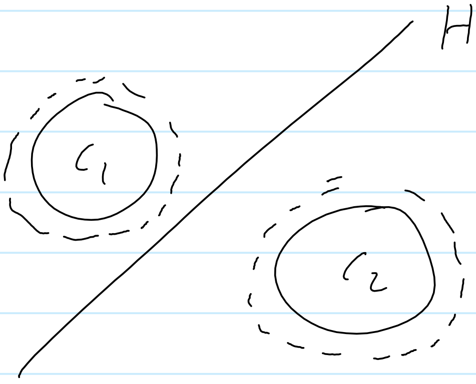
Strong Separation :

- H is said to separate C_1 & C_2 strongly if there exists some $\varepsilon > 0$ such that $C_1 + \varepsilon B$ is contained in one of the open half-spaces and $C_2 + \varepsilon B$ is contained in the opposite open half-space, when B is the unit-ball $\{x \mid \|x\| < 1\}$



- Both sets are strictly away from H.

- Strong separation

$\Longleftrightarrow$ There exists a vector b such that

$$\inf\{bx \mid x \in C_1\} > \sup\{bx \mid x \in C_2\}$$

Strict Seperation (skip)

- H is said to separate C_1 & C_2 strictly if both C_1 & C_2 belong to opposing open halfspaces.

- $\inf\{bx \mid x \in C_1\} \geq \sup\{bx \mid x \in C_2\}$

  & There exists a number a such that

  $bx < a$ for all $x \in C_1$

  $bx > a$ for all $x \in C_2$

Relative interior

Thursday, February 05, 2009 9:23 AM

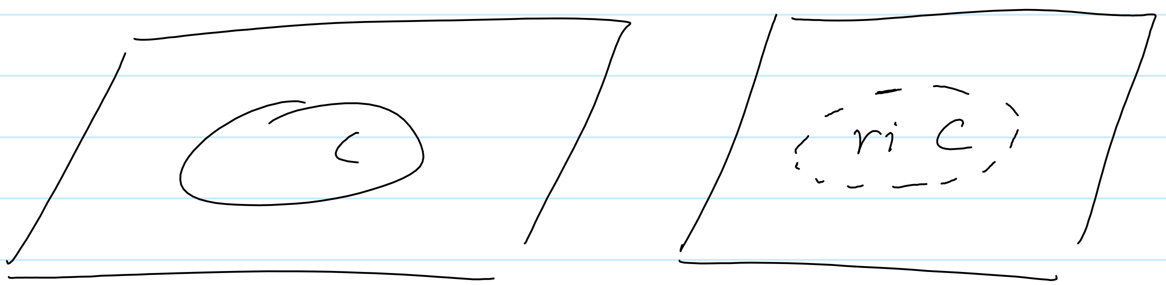
- A point x is in the relative interior of C if

$$\{y \mid \|x-y\| \leq r\} \cap \text{aff } C \subset C$$

for some $r > 0$

- $\text{ri } C = \{x \in C \mid B(x,r) \cap \text{aff } C \subset C \text{ for some } r>0\}$

- The relative interior is the interior relative to the affine hull of C



int $C = \emptyset$

- The relative boundary of a set C is defined as

$$\underbrace{\text{cl } C}_{\text{closure of } C} \setminus \text{rel int } C$$

Ⓠ What is the relative interior of a set containing one point?

Ⓐ $\operatorname{ri}\{A\} = \{A\}$

Since the affine hull of $\{A\}$ is $\{A\}$

Ⓠ A line segment?

Ⓠ A triangle?

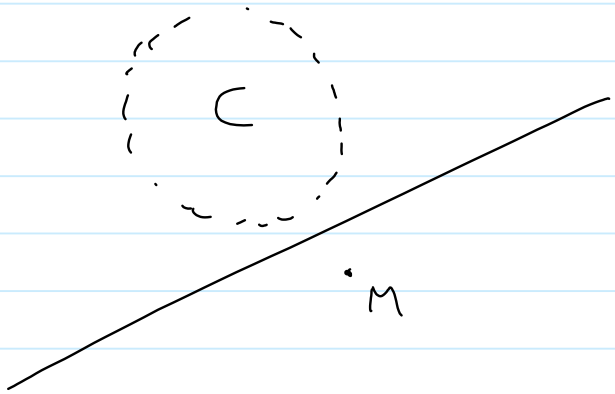
- For a non-empty convex set, its relative interior is always non-empty.

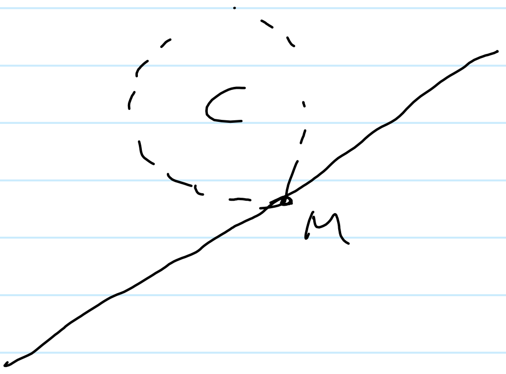
Separation theorem

Thursday, February 05, 2009 9:41 AM

- Thm 11.2 in Rockafellar

- Let C be a non-empty relatively open convex set in R^n, and let M be a non-empty affine set in R^n not meeting C (M could be a single point). Then, there exists a hyperplane H containing M, such that one of the open half-spaces associated with H contains C.





- In Rockafellar, this is the basis for other versions of the separation theorem
    - It is even stronger than proper separation (weaker than strong separation)
    - but very specialized.

- For us, we will use the special case when M is a point

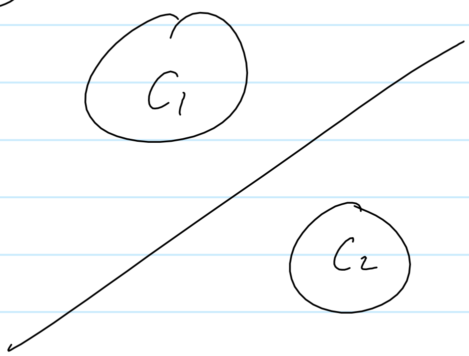
    - See convex combinations of infinite points.

- Thm 11.3 in Rockafellar

- Let C_1 & C_2 be non-empty convex sets in R^n. In order that there exists a hyperplane separating C_1 & C_2 <u>properly</u>, it is necessary and sufficient that $ri\, C_1$ & $ri\, C_2$ have no points in common. (They are disjoint).

Example:

1



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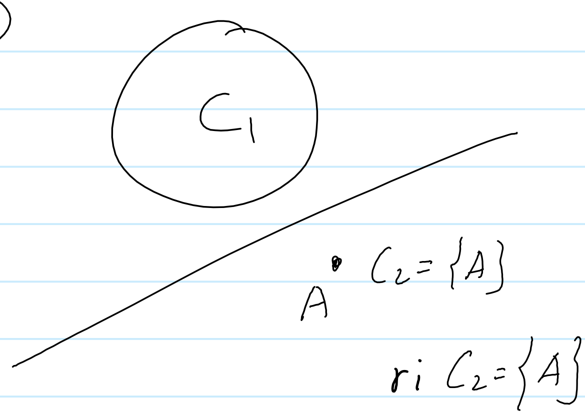

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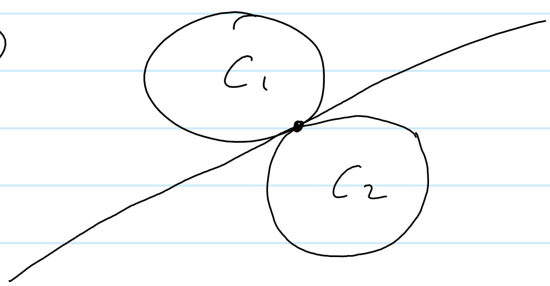

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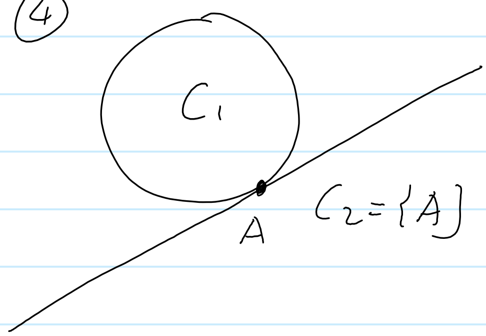

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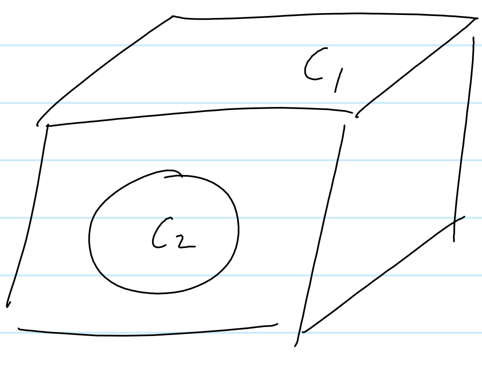


Unlike Thm 11.2, C_1 may have a part in H.

(Thm 11.4 in Rockafellar)

- Let C_1 & C_2 be non-empty convex sets in R^n. In order that there exists a hyperplane separating C_1 & C_2 strongly, it is necessary and sufficient that

$$\inf \left\{ \|x_1 - x_2\| \,\middle|\, x_1 \in C_1, x_2 \in C_2 \right\} > 0$$

i.e., C_1, C_2 are away by a positive distance

Convex combinations of countably infinite points

Wednesday, March 1, 2023 11:10 AM

- Let C be a convex set.
- $x_1, x_2, \cdots \in C$

  $\theta_1, \theta_2, \cdots \in (0,1)$ & $\sum_{i=1}^{+\infty} \theta_i = 1$

- Assume that the limit

$$y = \sum_{i=1}^{+\infty} \theta_i x_i$$

  exists

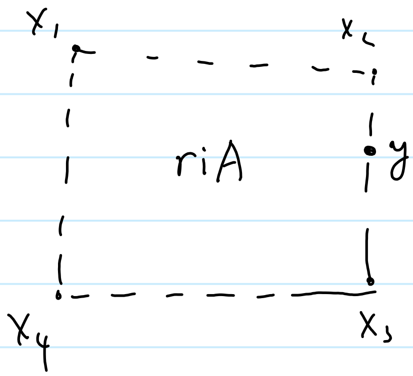
- Then we must have $y \in C$
  - More obvious when C is a closed set
  - But here we don't need C to be closed

Proof:

- Let A be the convex hull of $\{x_1, x_2, \cdots\}$
  - A is a convex set & $A \subset C$
- Further, $\mathrm{ri}\, A$ is always non-empty.
- We will show that $y \in \mathrm{ri}\, A$ by contradiction.

- We will show that $y \in \text{ri}A$ by contradiction.

- Suppose that $y \notin \text{ri}\,A$. Then by Thm 11.2, there exists a hyperplane that contains y, such that $\text{ri}A$ lies in one of the open half-spaces



- Since y is the convex combination of $\{x_1, x_2, \cdots\}$, it is definitely not outside $\text{cl}A$
- However, it seems unlikely that y is at the boundary either, because all x has positive weights
- unless all x are on the same hyperplane??

⇒ cannot separate by open half-space!

- This implies that there exists $b \in R^n$ such that

$$bx < by \text{ for all } x \in \text{ri}A \qquad (*)$$

- We must then have

$$bx \le by \quad \text{for all } x \in A.$$

- But

- But

$$y = \sum_{i=1}^{t+p} \theta_i x_i$$

$$\Rightarrow \quad \sigma = \sum_{i=1}^{t+p} \theta_i (b x_i - b y)$$

- We must then have

$$b x_i = b y \text{ for all } i$$

- But this implies that

$$b x = b y \text{ for all convex combinations of } \{x_1, x_2 \dots\}$$

  - including those x in $r_i A$.

  - This contradicts $(*)$.