Lec2

Wednesday, January 14, 2009 10:29 PM

- The first set of lectures will introduce
    - convex sets
    - convex functions
    - convex optimization problems minimize a convex function over a convex set
- Familiarity with these concepts are important because it helps us quickly recognize whether we are on the path of formulating a convex opt. problem.
- Let us build the knowledge towards convex sets first.

Sets (skipped)

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We often work with the space R^n.

Euclidean norm : $\|x\| = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}$

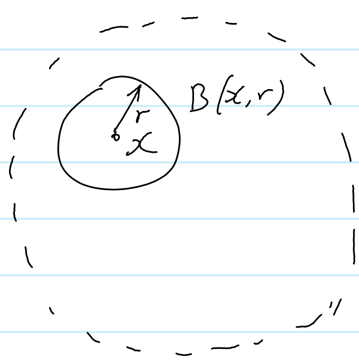
Open set O:

- no point in O is on the "border"
- More precisely, define a ball centered at x

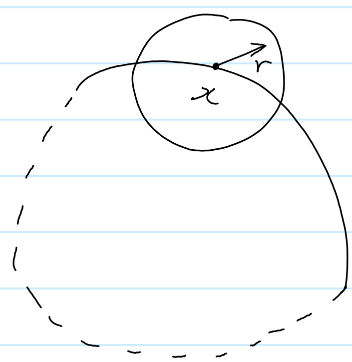
$$B(x,r) = \{y : \|y - x\| \le r\}$$

__Definition:__ A set O is an open set if for any $x \in O$, there exists $r > 0$ such that $B(x,r) \subset O$

no border



Open set O

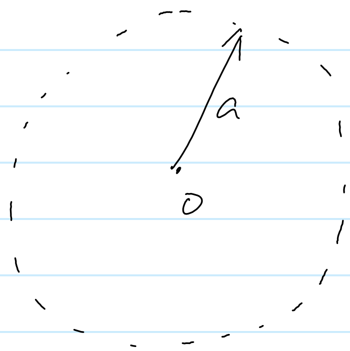


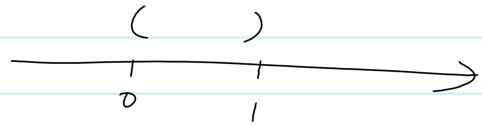
Not an open set.

Examples:

$\|x\| < a$ is an open set

$\|x\| > a$ is an open set

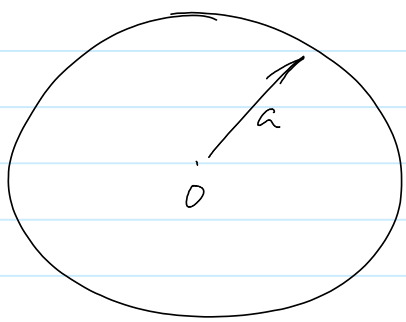




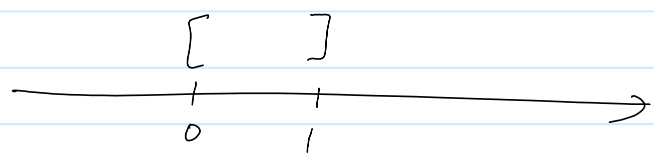
$(0,1)$ is an open set.

Closed set F: A set F is a closed set if $\overline{F}$ is an open set

- Since an open set does not contain the boundary, the complement will



Examples
- $\|x\| \le r$
  $\|x\| \ge r$
- $[0,1]$



- Any intersection of closed sets is still a closed set

- Any union of open sets is still an open set

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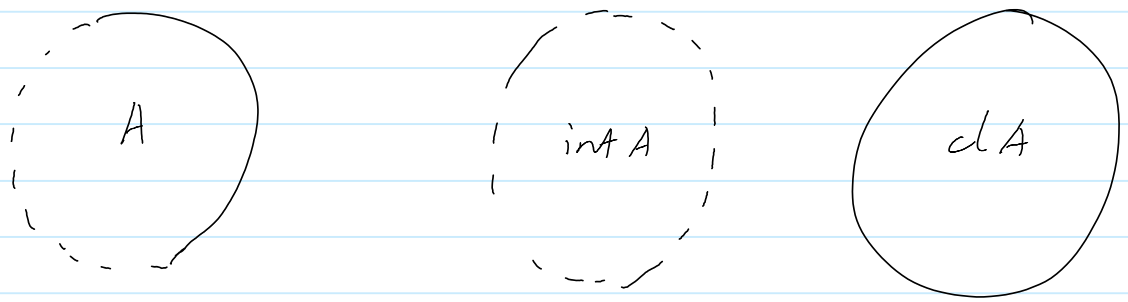
closure $\mathrm{cl}A$: smallest closed set containing A

interior $\text{int} A$: the largest open set contained in A.



boundary $\text{cl} A \setminus \text{int} A$.

$A = (0, 1]$ $\quad$ $\text{int} A = (0, 1)$ $\quad$ $\text{cl} A = [0, 1]$

boundary $\{0\} \cup \{1\}$

- Many important results of convex theory apply to both open & closed sets
- In some cases, convexity can replace the need to assume whether a set of closed or open (see later).

Affine set

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- Even though we are more interested in convex sets, there is some close relation between convex and affine sets
  - e.g. seperation theorem.
  - which are very important for convex theory
- Also, affine sets are convex sets.

We oten work with the space R^n.

A set C is an affine set if for any $x_1, x_2, \cdots, x_K \in C$, then

$$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_K x_K \in C$$

for all $\theta_1 + \theta_2 + \cdots + \theta_K = 1$

The linear combination

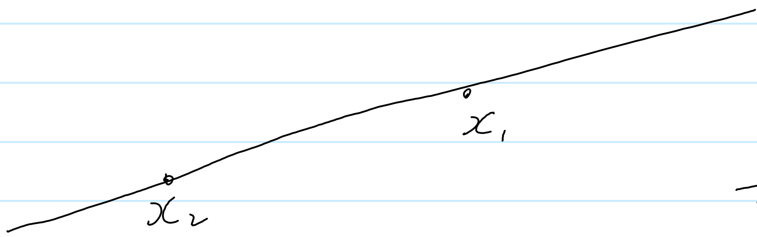
<u>Examples</u>

① A 1-dim affine set.

Suppose $x_1, x_2 \in C$. For C to be an affine set, we must have

$$\theta_1 x_1 + (1-\theta_1) x_2 \in C$$

$$\Leftrightarrow \quad x_2 + \theta_1 (x_1 - x_2) \in C$$

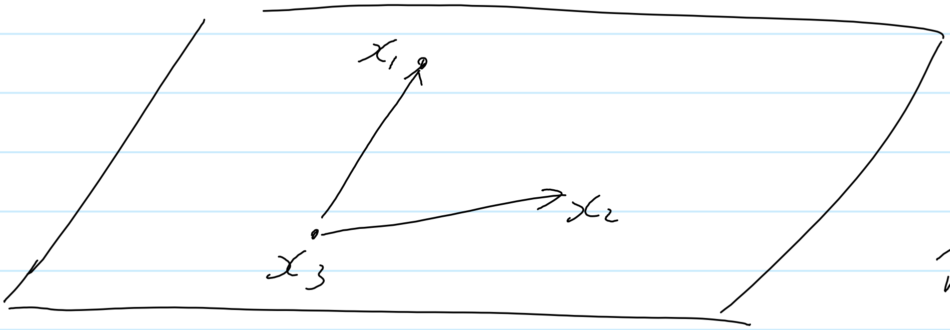


all points on the line that passes through x_1 & x_2

② A 2-dim affine set

Suppose $x_1, x_2, x_3 \in C$, then

$$\theta_1 x_1 + \theta_2 x_2 + (1-\theta_1-\theta_2) x_3 \in C$$
$$\Leftrightarrow \quad x_3 + \theta_1 (x_1 - x_3) + \theta_2 (x_2 - x_3) \in C$$



all points on a plane that passes through x_1, x_2, x_3

Assume that an affine C is a subset of R^3.
If there is another point $x_4 \in C$ that does not belong to the plane, then $C = R^3$.

- Thus, the 2-dim affine set is the largest affine set in R^3 that does not span the whole space
  $\Rightarrow$ In R^n, this will be called a hyperplane.

③ The set $\{x \mid Ax = b\}$ is an affine set

For any $x_1, \cdots x_K \in \{x \mid Ax = b\}$

$$\Rightarrow \quad Ax_1 = b, \; Ax_2 = b, \; \cdots, \; Ax_K = b$$

$$\Rightarrow \quad A(\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_K x_K)$$
$$= (\theta_1 + \theta_2 + \cdots \theta_K) b = b$$

for all $\theta_1 + \theta_2 + \cdots + \theta_k = 1$

- In R^3, the line will correspond to 2 equations
  the plane will correspond to 1 equation

  In R^n, 1 equation (i.e., $ax = b$) will be a hyper-plane.

Any affine set can be described by some linear equations (see later)

Affine hull

Friday, January 09, 2009 5:07 PM

Hull: the husk, shell, or outer covering of a seed or fruit.

Pasted from <http://dictionary.reference.com/browse/hull>

The affine hull of a set C is the smallest affine set that contains C. Denote it by $\text{aff } C$.

Ⓠ How to find the affine hull of C

Ⓐ Take any points in C. Their linear combination must belong to the affine hull.

Claim:

$$\text{aff } C = \left\{ y \;\middle|\; \begin{array}{l} y = \theta_1 x_1 + \cdots + \theta_k x_k, \; x_1, \cdots x_k \in C \\ \& \;\; \theta_1 + \cdots + \theta_k = 1 \end{array} \right\}$$

Let us skip the proof as a similar proof will be discussed for convex hulls.

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Subspace (brief)

Friday, January 09, 2009 5:34 PM

Closely related to affine sets is the concept of subspaces

- A subspace is an affine set that passes through the origin
- Any affine set is a subspace plus an offset

(skip)

A set V is a subspace if for any $x_1, x_2, \cdots x_k \in V$, then

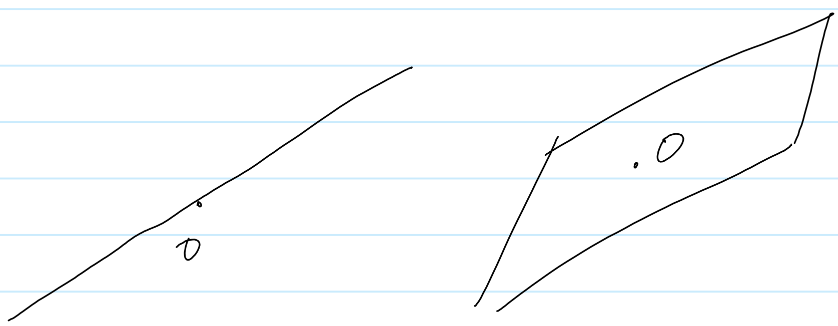
$$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in V$$

for any $\theta_1, \cdots, \theta_k$

Difference from affine sets

- We do not need $\theta_1 + \cdots + \theta_k = 1$
- The origin must belong to a subspace

keep



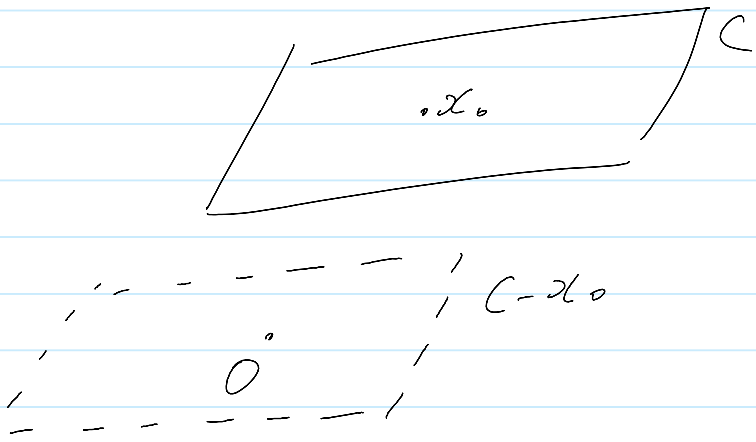
$\{x \mid Ax = 0\}$

skip

For any affine set C, pick a point $x_0 \in C$, Let

$$C - x_0 = \{x - x_0 \mid x \in C\}$$

Then $C - x_0$ is a subspace



Proof: Assume that $v_1, v_2, \cdots v_k \in C - x_0$

Then

$$v_1 = x_1 - x_0, \quad v_2 = x_2 - x_0, \quad \cdots \quad v_k = x_k - x_0$$

for some $x_1, x_2, \cdots x_k \in C$

Hence

$$\begin{aligned}
&\theta_1 v_1 + \theta_2 v_2 + \cdots + \theta_k v_k \\
&= (\theta_1 x_1 + \cdots \theta_k x_k) - (\theta_1 + \cdots + \theta_k) x_0 \\
&= \left[(\theta_1 x_1 + \cdots \theta_k x_k) + x_0 (1 - (\theta_1 + \cdots + \theta_k)) \right] - x_0
\end{aligned}$$

$\in C - x_0.$

#

Summary:

- Any affine set is a subspace plus an offset
- The subspace associated with the affine set C does not depend on the choice of x_0

- Often a subspace is the set of x such that

$$Ax = 0$$

Then the affine set is

$$Ax = b$$

⋮

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Convex sets

Friday, January 09, 2009 5:43 PM

A set C is convex if for any $x_1, \cdots, x_k \in C$ we must have

$$\theta_1 x_1 + \cdots + \theta_k x_k \in C$$

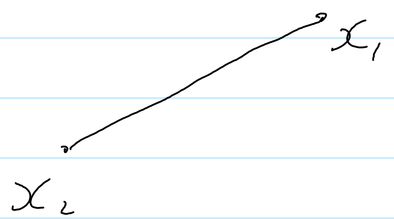
for all $\theta_1 + \theta_2 + \cdots + \theta_k = 1$ and $0 \le \theta_1, \cdots, \theta_k \le 1$.

convex combinations

Assume that $x_1, x_2 \in C$, for C to be a convex set we must have

$$\theta_1 x_1 + (1-\theta_1) x_2 \in C$$

$$\Leftrightarrow \quad x_2 + \theta_1 (x_1 - x_2) \in C \qquad 0 \le \theta_1 \le 1$$

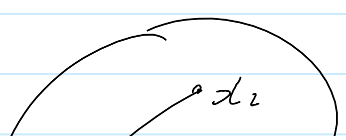


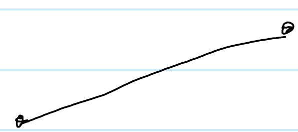
The line segment between x_1 & x_2

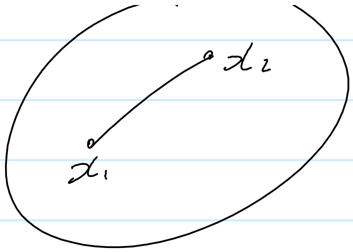
Rough speaking: a set is convex if you can go from every point to another point via a straight line ~~without~~ leaving the set.

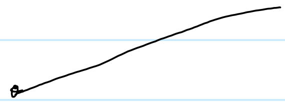
Difference from affine sets: we only need the line segment belongs to C. We don't need the part outside x_1 or x_2.

Examples

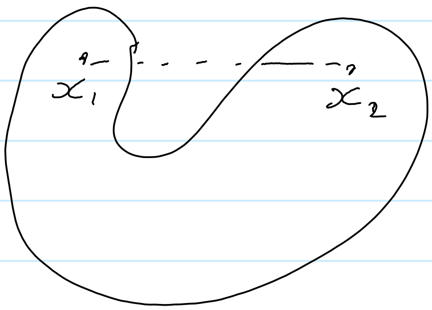








This is not a convex set



A convex set does not have holes.

Q Are affine sets convex?

Q Can a convex set be an open set? or a closed set?

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Convex hull

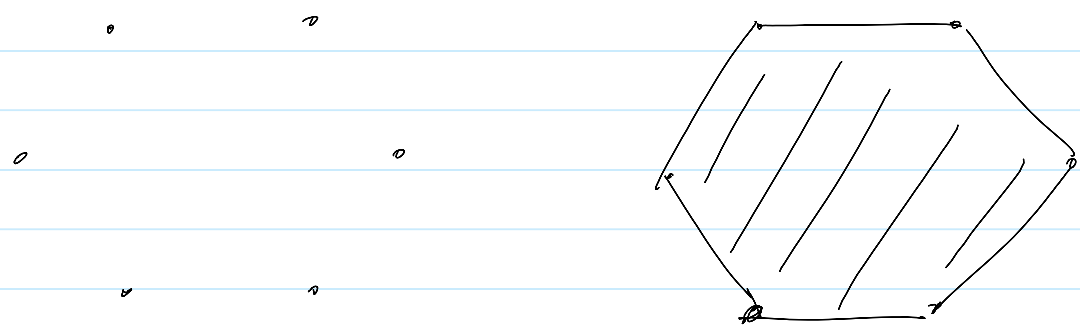
Friday, January 09, 2009 6:00 PM

The convex hull of a set C is the smallest convex set containing C.

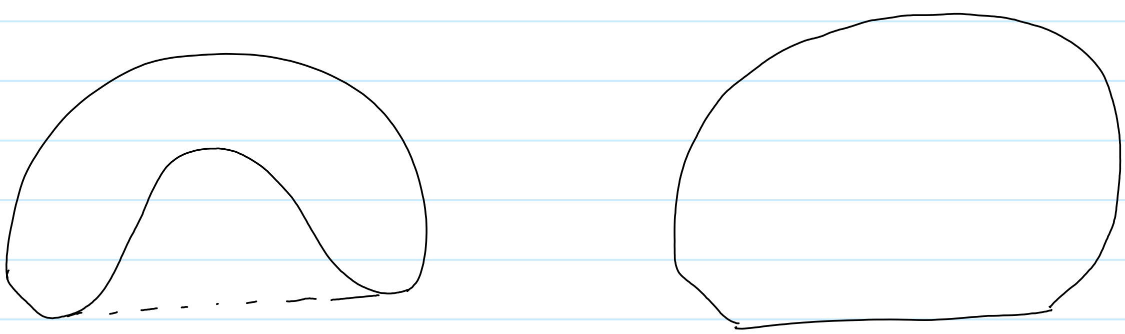
Take any two points $x_1, x_2 \in C$, then the line segment btw x_1 & x_2 must belong to the convex hull.

So we can construct convex hull in this way

Examples :

① 

$C = \{x_1, x_2, x_2, x_4, x_5, x_6\}$ Conv(C)

② 

C Conv C

Think of the region enclosed by a rubber-band.

Claim: $\operatorname{Conv} C = \{ y \mid y = \theta_1 x_1 + \cdots + \theta_k x_k$ for some $x_1, x_2, \cdots x_k \in C$, $0 \le \theta_1, \theta_2 \cdots \theta_k \le 1$, $\theta_1 + \cdots + \theta_k = 1 \}$

the set of all convex combinations of C

Denote the RHS as the set A.

(Q) Why is it the smallest convex set containing C?

(A) ① A is convex

② For any convex set B containing C, B must contain A.

To show ①: we need to show that, if $v_1, \cdots v_j$ are all convex combinations of elements of C, then any convex combinations of $v_1, \cdots, v_j$ is also a convex combinations of elements of C

To see this, assume

$$V_1 = \theta_1^1 x_1 + \cdots + \theta_k^1 x_k$$
$$V_2 = \theta_1^2 x_1 + \cdots + \theta_k^2 x_k$$
$$\vdots$$
$$V_j = \theta_1^j x_1 + \cdots + \theta_k^j x_k$$

For any $0 \le \gamma_1, \gamma_2, \cdots \gamma_j \le 1$ & $\gamma_1 + \gamma_2 + \cdots + \gamma_j = 1$, we have

$$\gamma_1 V_1 + \cdots + \gamma_j V_j$$
$$= x_1 (\gamma_1 \theta_1^1 + \gamma_2 \theta_1^2 + \cdots + \gamma_j \theta_1^j)$$

$$+ x_2 (\gamma_1 \theta_2^1 + \gamma_2 \theta_2^2 + \cdots + \gamma_j \theta_2^j)$$

$$\vdots$$

$$+ x_k (\gamma_1 \theta_k^1 + \gamma_2 \theta_k^2 + \cdots + \gamma_j \theta_k^j)$$

They are btw 0 & 1.
and their sum is equal to
$\gamma_1 + \gamma_2 + \cdots + \gamma_j = 1$ #

To show part (2): trivially from definition of the convex hull.

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Brief summary

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The beauty of the theory of convexity is that every notion has both an algebraic definition & a geometric interpretation

It is important to be able to go back and forth.

Algebraic forms

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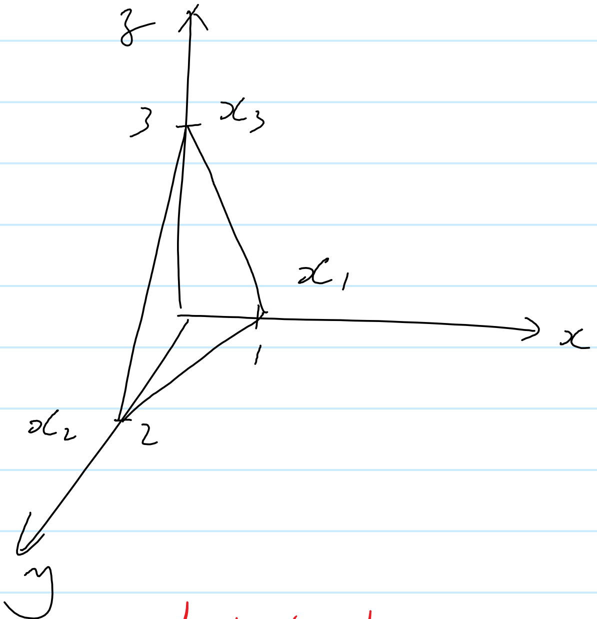
The above concepts are useful when we need to write down algebraic forms of the convex hull



Q What is the convex hull of the set containing the three points?

A It must be a triangle

The plane (subspace) that this triangle belongs to is

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$$

Need constraints $x, y, z \geq 0$

Think of this as the transmission rate to each user individually.

Alternately, the convex hull is the set of w's such that

$$w = \theta_1 (1,0,0) + \theta_2 (0,2,0) + \theta_3 (0,0,3)$$

$$0 \leq \theta_1, \theta_2, \theta_3 \leq 1$$

$$\theta_1 + \theta_2 + \theta_3 = 1$$

Can show that the two descriptions are equivalent.

$$\Leftrightarrow \quad w = (\theta_1, 2\theta_2, 3\theta_3)$$

JD

Infinite convex combinations and expectation

Friday, January 09, 2009 6:20 PM

Although the definition of a convex set only uses finite convex combinations, the property holds under more general settings

(1) Suppose C is convex. An infinite sequence of points $x_1, x_2, \cdots \in C$. For any sequence of non-negative numbers $\theta_1, \theta_2, \cdots$ such that

$$\sum_{i=1}^{+\infty} \theta_i = 1$$

we must have

$$\sum_{i=1}^{+\infty} \theta_i x_i \in C$$

whenever the summation converges

Why is this not obvious?

- We can write $\sum_{i=1}^{+\infty} \theta_i x_i$ as the limit of a sequence

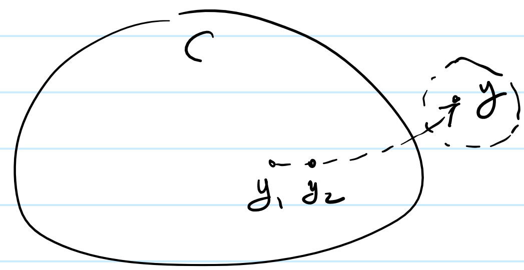
$$y_1 = x_1$$
$$y_2 = \theta_1 x_1 + (1-\theta_1) x_2$$
$$y_3 = \theta_1 x_1 + \theta_2 x_2 + (1-\theta_1-\theta_2) x_3$$
$$\vdots$$
$$y_n = \sum_{i=1}^{n-1} \theta_i x_i + \left[1 - \sum_{i=1}^{n-1} \theta_i\right] x_n$$

- When C is convex, we know that each $y_n \in C$

- If C is in addition closed, then $\lim_{n \to +\infty} y_n \in C$

  - Why? If not, then $y \triangleq \lim_{n \to +\infty} y_n \in \overline{C}$ which is an open set.

an open set.



- Then there must be a ball around y that is entirely in $\overline{C}$
- But that cannot happen since $y_1 \cdots y_n \in C$ and they are closer and closer to y!

- The unusual part is, even if C is NOT closed, the above conclusion is still true!

$\Rightarrow$ Convexity carries important topological properties.

(2) Suppose a random variable X is chosen from a convex set C according to some probability distribution p.

Then $E(X) = \int_C x \, p_x \in C$

We skip the proof. Need to use the fact that any convex set is the intersections of halfspaces & hyperplanes.

JS