Lec18-new

Sunday, February 26, 2023 10:00 AM

I will travel next week. Haoqian (our TA) will give the lectures next week.

Complementary slackness

Sunday, February 15, 2009 8:56 AM

- We have seen some important properties of the pair of primal solutions x^* & dual solutions (λ^*, ν^*)
- If duality gap is zero, then

① $f_i(x^*) \le 0$, $h(x^*) = 0$
② $\lambda^* \ge 0$
③ x^* minimizes $L(x, \lambda^*, \nu^*)$ over x

Below the 4th important property.

Statement: Let λ^*, ν^* be the optimal solution to the dual & x^* be the optimal solution to the primal. If the duality gap is zero, then

$$\lambda^* f_i(x^*) = 0 \quad \forall i$$

Further, x^* minimize $L(x, \lambda^*, \nu^*)$.

- If $\lambda_i^* > 0$, then $f_i(x^*) = 0$

  If $f_i(x^*) < 0$, then $\lambda_i^* = 0$

- If the Lagrange multiplier is non-zero, then the constraint must be binding/tight.

Proof: Assume $g(\lambda^*, \nu^*) = f_0(x^*)$.

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Since

$$g(\lambda^*, \nu^*) \le L(x^*, \lambda^*, \nu^*)$$
$$= f_0(x^*) + \sum \lambda_i^* f_i(x^*) + \sum \nu_i^* \underbrace{h_i(x^*)}_{=0}$$

$$\Rightarrow \quad 0 \le \sum \underbrace{\lambda_i^*}_{\ge 0} \underbrace{f_i(x^*)}_{\le 0}$$

$$\Rightarrow \quad \lambda_i^* f_i(x^*) = 0 \quad \forall i$$

Further, $g(\lambda^*, \nu^*) = L(x^*, \lambda^*, \nu^*) = f_0(x^*)$. #

Important Remarks:

- No convexity is assumed.

- This holds when we pair any primal solution x^* with any dual solution λ^*!

These properties are useful when we want to find the primal solution after we found the dual solution.

- If the primal variable that minimizes $L(x, \lambda^*, \nu^*)$ is unique. → Great!

- If NOT, we know as a fact that x^* minimizes $L(x, \lambda^*, \nu^*)$

  - But, does it mean that any $\bar{x}$ that minimizes $L(x, \lambda^*, \nu^*)$ is that minimizes $L(x, \lambda^*, \nu^*)$ is optimal?

- The answer is NO!

- First, such an $\bar{x}$ may violate the primal constraints.

- Second, even if $\bar{x}$ satisfies the primal constraints, we only have

$$f_0(\bar{x}) + \sum_i \lambda_i^* f_i(\bar{x}) + \sum_i \nu_i^* h_i(\bar{x})$$

$$= f_0(x^*) + \underbrace{\sum_i \lambda_i^* f_i(x^*)}_{0} + \sum_i \nu_i^* h_i(\bar{x})$$

$$= f_0(x^*).$$

- But $\lambda_i^* f_i(\bar{x})$ may be < 0

$$\Rightarrow f_0(\bar{x}) \geq f_0(x^*)$$

- However, if in addition we know that $\bar{x}$ also satisfies the CS condition, i.e.,

$$\lambda_i^* f_i(\bar{x}) = 0 \quad \forall i$$

Then $f_0(\bar{x}) = f_0(x^*)$.

- In summary, find $\bar{x}$ that satisfies the primal constraints and the

the primal constraints and the
cs - condition.

- That $\bar{x}$ must be optimal!

- if $\lambda_i^* > 0$, the corresponding constraint must be binding/tight.

  $\Rightarrow$ helps to solve $\bar{x}$.

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KKT condition

Sunday, February 15, 2009 8:59 AM

KKT (Karush-Kuhn-Tucker) Conditions

A pair of primal-dual points (x^*, λ^*, ν^*) are said to satisfy the KKT condition if

- $f_i(x^*) \le 0$, $h_i(x^*) = 0$ (satisfy the constraints)
- $\lambda_i^* \ge 0$ (valid Lagrange multiplier)
- $\lambda_i^* f_i(x^*) = 0 \quad \forall i$ (complementary slackness)
- x^* minimizes $f_0(x) + \sum_i \lambda_i^* f_i(x) + \sum_i \nu_i^* h_i(x)$ over all x. (min Lagrangian)

Note: If the functions are differentiable & convex, then the last condition is equivalent to

$$\nabla f_0(x^*) + \sum_i \lambda_i^* \nabla f_i(x^*) + \sum_i \nu_i^* \nabla h_i(x^*) = 0 \quad (*)$$

KKT condition — Zero duality gap.

primal-dual optimality

(1) If the duality gap is zero, then any

pair of primal & dual optimal points must satisfy the KKT condition.

$$\text{KKT} \Longleftarrow \begin{array}{c}\text{Zero duality-gap} \\ + \\ \text{primal \& dual optimal}\end{array}$$

- No convexity is assumed.
- follows from the complementary slackness condition.

② If a pair of primal & dual points satisfy the KKT condition, then they must be optimal & there is zero duality gap.

$$\text{KKT} \Longrightarrow \begin{array}{c}\text{zero duality-gap} \\ \text{primal-dual optimal}\end{array}$$

We always have, for any feasible x & $\lambda \geq 0$:

$$g(\lambda, \nu) \leq L(x, \lambda, \nu) \leq f_0(x)$$

If the KKT condition holds, then

$$g(\lambda^*, \nu^*) = L(x^*, \lambda^*, \nu^*) = f_0(x^*)$$

Hence, the duality gap is zero & x^* and (λ^*, ν^*) are the primal & dual optimal points, respectively.

- Convexity is needed only for the first-order condition (*). (Otherwise (*) may not imply that x^* minimizes the Lagrangian).

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(3) If the Slater condition holds, and the primal problem is convex, then the dual optimal solution always exist, and the duality gap is zero.

- Hence, the KKT condition provides a necessary & sufficient condition for optimality with constraints.

$$\text{KKT} \xLeftrightarrow{\text{convex + Slater}} \text{primal-dual optimal}$$

- Further, the 4th condition in KKT is simplified to a gradient condition.
  - For non-convex problems, however, this simplified form is necessary but not sufficient.

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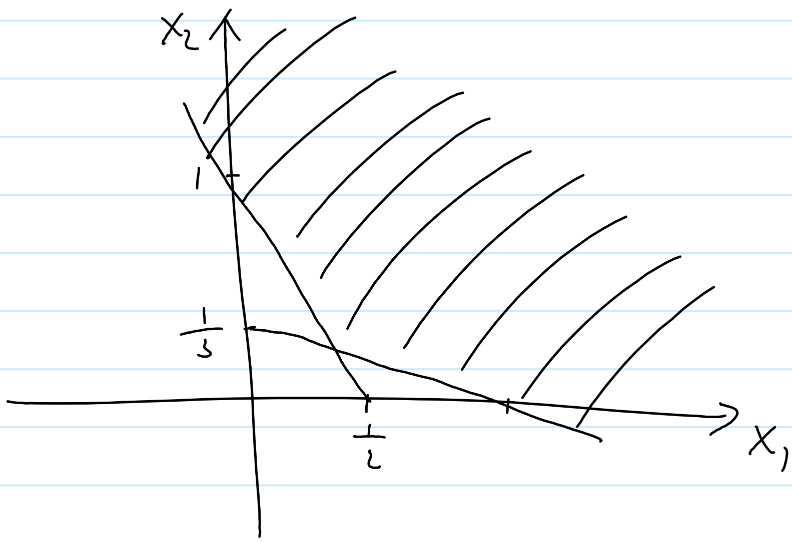
Example: KKT

Tuesday, February 22, 2011 8:06 PM

We have studied the dual of this problem:

$$\min \quad x_1^2 + 9x_2^2$$
$$\text{sub to} \quad 2x_1 + x_2 \geq 1 \qquad \lambda_1$$
$$x_1 + 3x_2 \geq 1 \qquad \lambda_2$$



Alternatively, we can directly use the KKT condition:

① $2X_1 + X_2 \geq 1, \quad X_1 + 3X_2 \geq 1$

② $\lambda_1 \geq 0, \quad \lambda_2 \geq 0$

③ $\lambda_1(2X_1 + X_2 - 1) = 0, \quad \lambda_2(X_1 + 3X_2) = 0$

④ (X_1, X_2) is the minimizer for $L(x_1, x_2, \lambda_1, \lambda_2)$

Start from ④

The Lagrangian is

$$L(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + 9x_2^2 + \lambda_1(1 - 2x_1 - x_2) + \lambda_2(1 - x_1 - 3x_2)$$

$$= x_1^2 - (2\lambda_1 + \lambda_2)x_1 + 9x_2^2 - (\lambda_1 + 3\lambda_2)x_2 + \lambda_1 + \lambda_2$$

Minimize the Lagrangian over x_1 & x_2

$$\frac{\partial L}{\partial x_1} = 2x_1 - (2\lambda_1 + \lambda_2) = 0$$

$$\frac{\partial L}{\partial x_2} = 18x_2 - (\lambda_1 + 3\lambda_2) = 0$$

$$\Rightarrow \quad x_1 = \frac{2\lambda_1 + \lambda_2}{2}, \quad x_2 = \frac{\lambda_1 + 3\lambda_2}{18}$$

Then consider several cases based on ③ & Solve x^*, λ^*, ν^*

- And check if the resulting solutions x^* & (λ^*, ν^*) meet both primal & dual constraints

Case 1: $\lambda_1 > 0, \lambda_2 > 0$

$$\Rightarrow \begin{cases} 2x_1 + x_2 = 1 \\ x_1 + 3x_2 = 1 \end{cases}$$

$$\Rightarrow \quad 5x_1 = 2 \Rightarrow x_1 = \frac{2}{5} \qquad x_2 = \frac{1}{5}$$

Check the dual variable

$$\begin{cases} \frac{2\lambda_1 + \lambda_2}{2} = \frac{2}{5} \\ \frac{\lambda_1 + 3\lambda_2}{18} = \frac{1}{5} \end{cases}$$

$$\Rightarrow \frac{5}{6}\lambda_1 = -\frac{1}{5} \Rightarrow \lambda_1 = -\frac{6}{25} < 0$$

Case 2: $\lambda_1 > 0$, $\lambda_2 = 0$

$$\Rightarrow \begin{cases} 2x_1 + x_2 = 1 \\ \lambda_2 = 0 \\ x_1 = \frac{2\lambda_1 + \lambda_2}{2} = \lambda_1 \end{cases}$$

$$x_2 = \frac{\lambda_1 + 3\lambda_2}{18} = \frac{\lambda_1}{18}$$

$$\Rightarrow \quad \frac{37}{18}\lambda_1 = 1 \qquad \lambda_1 = \frac{18}{37}$$

$$x_1 = \frac{18}{37}$$

$$x_2 = \frac{1}{37}$$

Check the primal constraint

$$x_1 + 3x_2 = \frac{21}{37} < 1$$

Case 3: $\lambda_1 = 0$, $\lambda_2 > 0$

$$\Rightarrow \begin{cases} x_1 + 3x_2 = 1 \\ \lambda_1 = 0 \\ x_1 = \frac{2\lambda_1 + \lambda_2}{2} = \frac{\lambda_2}{2} \\ x_2 = \frac{\lambda_1 + 3\lambda_2}{18} = \frac{\lambda_2}{6} \end{cases}$$

$$\Rightarrow \quad \lambda_2 = 1, \quad \lambda_1 = 0$$

$$x_1 = \frac{1}{2}, \quad x_2 = \frac{1}{6}$$

Check primal constraint

$$2x_1 + x_2 = \frac{7}{6} > 1.$$

Hence, this is the optimal solution.

Case 4: $\lambda_1 = 0$, $\lambda_2 = 0$

$$\Rightarrow \quad x_1 = 0, \quad x_2 = 0$$

This violates the primal constraints.