Lec17-mwf

Thursday, February 24, 2011 10:07 PM

Partial Lagrangian

Wednesday, February 04, 2009 10:27 AM

- We also have the flexibility to relax some constraint (by Lagrange multipliers) but not others

$$\min \ f_0(x)$$
$$\text{sub to} \quad f_1(x) \le 0 \qquad \lambda$$
$$f_2(x) \le 0 \qquad \Leftarrow \text{do not want to relax}$$

$$L(x,\lambda) = f_0(x) + \lambda f_1(x)$$

$$g(\lambda) = \min \ L(x,\lambda) \quad \text{sub to} \quad f_2(x) \le 0$$

Dual: $$\max \ g(\lambda) \quad \text{sub to} \quad \lambda \ge 0$$

- Reason ~~this~~ is okay?
  - Think of the objective func. as

$$f_0'(x) = \begin{cases} f_0(x) & \text{if } f_2(x) \le 0 \\ +\infty & \text{o/w.} \end{cases}$$

This is useful when some constraints are "easy" or not coupled.

- $x_i \geq 0$

60

Linear program

Sunday, February 01, 2009 2:00 PM

$$\begin{aligned} \min \quad & c^T x \\ \text{sub to} \quad & Ax = b \qquad \nu \\ & x \geq 0 \qquad \lambda \end{aligned}$$

$$\Leftrightarrow \quad -x \leq 0$$

$$L(x, \lambda, \nu) = c^T x - \lambda^T x + \nu^T (Ax - b)$$

$$= (c^T - \lambda^T + \nu^T A) x - \nu^T b$$

$$g(\lambda, \nu) = \min_x L(x, \lambda, \nu)$$

$$= \begin{cases} -\nu^T b & \text{if} \quad c - \lambda + A^T \nu = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Note that $g(\lambda, \nu)$ is linear on a convex set

$$\Rightarrow \qquad \max g(\lambda, \nu)$$

$$\Leftrightarrow \qquad \max_{\lambda \geq 0} \; -b^T \nu$$

$$\text{sub to} \quad c - \lambda + A^T \nu = 0$$

Alternatively, only use one Lagrange multiplier ν

$$L(x, \nu) = c^T x + \nu^T (Ax - b)$$

$$= (c^T + \nu^T A) x - \nu^T b.$$

Now we should

$$\min \; L(x, \nu)$$
$$\text{sub to} \quad x \geq 0$$

$$\Rightarrow \text{If } (c^T + \nu^T A)_i \geq 0 \Rightarrow x_i = 0$$
$$(c^T + \nu^T A)_i < 0 \Rightarrow x_i \to +\infty$$

$$\Rightarrow \quad g(\nu) = \min_{x \geq 0} L(x, \nu)$$
$$= \begin{cases} -\nu^T b & \text{if } (c^T + \nu^T A)_i \geq 0 \text{ for all } i \\ -\infty & \text{o/w.} \end{cases}$$

Therefore, the dual problem is

$$\max_{\nu} \; -\nu^T b$$
$$\text{sub to} \quad c^T + \nu^T A \geq 0$$

50

Weak and strong duality

Wednesday, February 04, 2009 10:31 AM

We have seen that:

① $g(\lambda, \nu)$ is always concave

② $d^* = \max_{\lambda \geq 0} g(\lambda, \nu) \leq p^*$ ⇐ optimal value of the primal problem

- $d^* - p^*$ is refered to as the "duality gap"

Ⓠ When is the duality gap zero?

- under fairly general assumptions, the duality gap will be zero if the primal problem is convex.

- Recall $D \triangleq \text{dom} f_0 \cap \text{dom} f_i \cap \text{dom} h_i$

Roughly speaking, we just need a feasible point in the "interior" of the constraints ("strictly inside")

- However, the "interior" of D or C may be $\emptyset$! (due to linear/equality constraints → hyperplane)

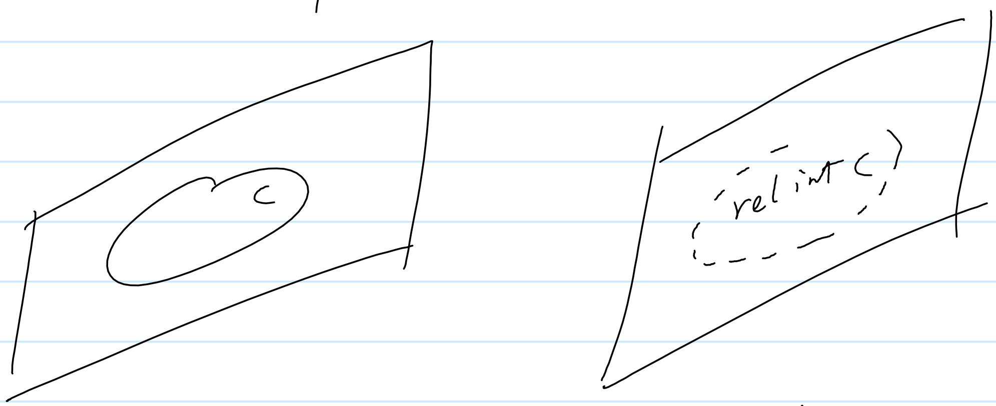
$\Rightarrow$ relative interior.

Relative Interior: (Boyd p23)

$$\text{rel int } C = \{x \in C \mid B(x,r) \cap \text{aff } C \subseteq C \text{ for some } r > 0\}$$

- Compare with the definition of interior

$$\text{int } C = \{x \in C \mid B(x,r) \subseteq C \text{ for some } r > 0\}$$



- int $C = \phi$ — rel int $C \neq \phi$

Slater Condition: There exists an $x \in$ rel int D such that

$$f_i(x) < 0 \quad \forall i$$
$$Ax = b$$

Theorem: (Boyd p 227) If the Slater condition holds, and in addition, rank $A = p$, where p is the number of equality constraints, then the duality gap is zero for the convex program $(*)$.

$\Rightarrow$ "strong duality"

Note: The slater condition is very mild.

- It only requires at least one feasible point in the relative interior of C

- Often the existence of such a point is trivial

Example: Rate control

$$\max \sum_s U_s(x_s)$$

$$\text{sub to} \quad \sum_s H_s^l x_s \le R^l \quad \forall l.$$

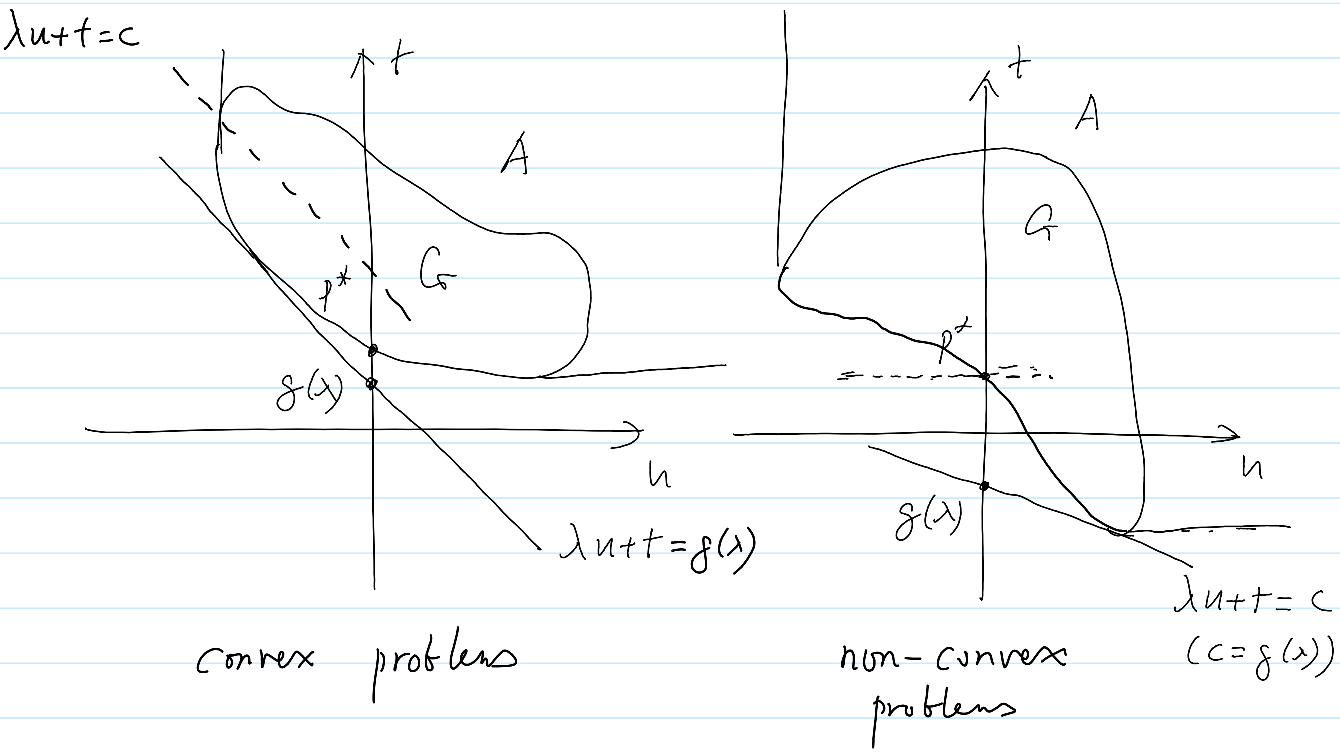
- need $R_l > 0$ for slater condition to hold.

(10)

Graphical interpretation

Sunday, February 01, 2009 1:44 PM

- Consider the case with only inequality constraints
- Boyd p234.
- Let $G = \{ \underbrace{f_1(x), \dots, f_m(x)}_{u'}, \underbrace{f_0(x)}_{t'} \mid x \in D \}$
- Let $A = \{ (u, t) \mid \exists x \in D, f_i(x) \le u_i, f_0(x) \le t \}$
  - like an epigraph, but without the variable x.
- If the problem is convex, then A is a convex set. (G may not be convex.)
  - Further, the lower boundary must be non-increasing in u.



- Let p^* denote the optimal value of

the primal problem:

Then $p^* = \inf\{t' \mid (u',t') \in G,\ u' \leq 0\}$

$= \inf\{t \mid (u,t) \in A,\ u = 0\}$

The Dual

- Let

$$g(\lambda) = \inf\{\lambda u' + t' \mid (u',t') \in G\}$$

$$= \inf\{\lambda u + t \mid (u,t) \in A\}, \quad \lambda \geq 0$$

- It is the intersection with the vertical-axis at the highest line below A.

- Clearly $p^* \geq g(\lambda)$

- If there is a hyperplane that supports A at the point $(0, p^*)$

Then $p^* = g(\lambda^*)$

- The slater condition ensures that there is some part of A on the left of the vertical axis $\Rightarrow$ supporting hyperplane must exist.

  - If the Slater condition does not hold:

- To argue about this, we need an important result for convex sets, called separation theorem.
  - It is quite technical, so we will focus on the implication of strong duality first, and return to separation theorem later.

25