IERG 6120: Homework 5

Due: October 30th, 2023 in class

1. (Smoothness condition of the dual) Let

$$g(\lambda) = \min_{x} U(x) + \lambda(x - R) \tag{1}$$

where $U(x)$ is a convex, twice-differentiable function of x such that $U''(x) \geq M > 0$ for all x. Note that $g(\lambda)$ is the dual objective function of the (primal) problem that minimizes $U(x)$ over the constraint $x \leq R$. Let $\partial g(\lambda)$ denote a subgradient of $g(\cdot)$ at λ.

   (a) Show that $g(\lambda)$ is a concave function of λ.

   (b) Show that

$$-[\partial g(\lambda_1) - \partial g(\lambda_2)]^T(\lambda_1 - \lambda_2) \geq M||\partial g(\lambda_1) - \partial(\lambda_2)||^2$$

   for all λ_1 and λ_2. (Hint: Express $\partial g(\lambda)$ by the point $x_0(\lambda)$ that minimizes $U(x) + \lambda(x - R)$.)

   (c) From Part (b), show that $\partial g(\lambda)$ satisfies a Lipschitz condition.

2. *Optimal power and bandwidth allocation in a Gaussian broadcast channel.* We consider a communication system in which a central node transmits messages to n receivers. Each receiver is assigned a certain amount of bandwidth $W_i \geq 0$, orthogonal to all other receivers. Further, the central node can assign a transmission power level $P_i \geq 0$ to each receiver. The power and bandwidth of a receiver channel determine its bit rate R_i via

$$R_i = \alpha_i W_i \log(1 + \beta_i P_i / W_i),$$

where α_i and β_i are known positive constants. (Note that W_i appears twice in the above expression. This is because, while the rate R_i is proportional to the bandwidth, the noise is also proportional to the bandwidth. Thus, the SNR is inversely proportional to W_i.) For $W_i = 0$, we take $R_i = 0$ (which is what you get if you take the limit as $W_i \to 0$).

The powers must satisfy a total power constraint, which has the form

$$\sum_{i=1}^{n} P_i \leq P_0,$$

where $P_0 \geq 0$ is a given total power available at the central node to be allocated among the receiver channels. Similarly, the bandwidths must satisfy

$$\sum_{i=1}^{n} W_i = W_0,$$

where $W_0 \geq 0$ is the given total available bandwidth. The optimization variables in this problem are the powers P_i and the bandwidths W_i.

The objective at the central node is to maximize the total sum-rate,

$$\sum_{i=1}^{n} R_i.$$

(a) Pose this problem as a convex optimization problem. You should justify why it is a convex problem.

(b) Associate a Lagrange multiplier λ for the constraint $\sum P_i \leq P_0$. (Do NOT associate a Lagrange multiplier for the constraint $\sum W_i = W_0$.) Write down the conditions to optimize the Lagrangian for a given λ.

(c) Using the condition in Part (b), show that the optimal power and bandwidth allocation must satisfy the following property: there must exist a positive number λ such that

$$P_i = \begin{cases} \frac{W_i}{\beta_i}(\frac{\alpha_i \beta_i}{\lambda} - 1) & \text{if } \frac{\alpha_i \beta_i}{\lambda} > 1 \\ 0 & \text{otherwise} \end{cases}.$$

Further, substitute this value of P_i into the condition in Part (b). Show that among the subset J of receivers with $\frac{\alpha_i \beta_i}{\lambda} > 1$, only those receivers i with

$$\alpha_i \log(\frac{\alpha_i \beta_i}{\lambda}) - \frac{\lambda}{\beta_i}(\frac{\alpha_i \beta_i}{\lambda} - 1) = \max_{j \in J} \left\{ \alpha_j \log(\frac{\alpha_j \beta_j}{\lambda}) - \frac{\lambda}{\beta_j}(\frac{\alpha_j \beta_j}{\lambda} - 1) \right\}$$

will be allocated non-zero bandwidth W_i.

(d) Develop an algorithm to iteratively compute the optimal value of λ. Describe how to adaptively control P_i and W_i based on this iterative algorithm.

3. Problems 5.1, 5.21, 5.27, 5.28 in Boyd and Vandenberghe.

Note: Please also start working on the rest of the simulation project. (You don't need to submit results with this homework yet.)