## IERG 6120: Homework 5

Due: November 6th, 2024 in class

1. (Smoothness condition of the dual) Let

$$g(\lambda) = \min_{x} U(x) + \lambda(x - R) \tag{1}$$

where U(x) is a convex, twice-differentiable function of x such that  $U''(x) \ge M > 0$ for all x. Note that  $g(\lambda)$  is the dual objective function of the (primal) problem that minimizes U(x) over the constraint  $x \le R$ . Let  $\partial g(\lambda)$  denote a subgradient of  $g(\cdot)$  at  $\lambda$ .

- (a) Show that  $g(\lambda)$  is a concave function of  $\lambda$ .
- (b) Show that

$$-[\partial g(\lambda_1) - \partial g(\lambda_2)]^T(\lambda_1 - \lambda_2) \ge M ||\partial g(\lambda_1) - \partial(\lambda_2)||^2$$

for all  $\lambda_1$  and  $\lambda_2$ . (Hint: Express  $\partial g(\lambda)$  by the point  $x_0(\lambda)$  that minimizes  $U(x) + \lambda(x-R)$ .)

- (c) From Part (b), show that  $\partial g(\lambda)$  satisfies a Lipschitz condition.
- 2. Optimal power and bandwidth allocation in a Gaussian broadcast channel. We consider a communication system in which a central node transmits messages to n receivers. Each receiver is assigned a certain amount of bandwidth  $W_i \ge 0$ , orthogonal to all other receivers. Further, the central node can assign a transmission power level  $P_i \ge 0$ to each receiver. The power and bandwidth of a receiver channel determine its bit rate  $R_i$  via

$$R_i = \alpha_i W_i \log(1 + \beta_i P_i / W_i),$$

where  $\alpha_i$  and  $\beta_i$  are known positive constants. (Note that  $W_i$  appears twice in the above expression. This is because, while the rate  $R_i$  is proportional to the bandwidth, the noise is also proportional to the bandwidth. Thus, the SNR is inversely proportional to  $W_i$ .) For  $W_i = 0$ , we take  $R_i = 0$  (which is what you get if you take the limit as  $W_i \to 0$ ).

The powers must satisfy a total power constraint, which has the form

$$\sum_{i=1}^{n} P_i \le P_0,$$

where  $P_0 \ge 0$  is a given total power available at the central node to be allocated among the receiver channels. Similarly, the bandwidths must satisfy

$$\sum_{i=1}^{n} W_i = W_0,$$

where  $W_0 \ge 0$  is the given total available bandwidth. The optimization variables in this problem are the powers  $P_i$  and the bandwidths  $W_i$ .

The objective at the central node is to maximize the total sum-rate,

$$\sum_{i=1}^{n} R_i.$$

- (a) Pose this problem as a convex optimization problem. You should justify why it is a convex problem.
- (b) Associate a Lagrange multiplier  $\lambda$  for the constraint  $\sum P_i \leq P_0$ . (Do NOT associate a Lagrange multiplier for the constraint  $\sum W_i = W_0$ .) Write down the conditions to optimize the Lagrangian for a given  $\lambda$ .
- (c) Using the condition in Part (b), show that the optimal power and bandwidth allocation must satisfy the following property: there must exist a positive number  $\lambda$  such that

$$P_i = \begin{cases} \frac{W_i}{\beta_i} \left(\frac{\alpha_i \beta_i}{\lambda} - 1\right) & \text{if } \frac{\alpha_i \beta_i}{\lambda} > 1\\ 0 & \text{otherwise} \end{cases}$$

Further, substitute this value of  $P_i$  into the condition in Part (b). Show that among the subset J of receivers with  $\frac{\alpha_i \beta_i}{\lambda} > 1$ , only those receivers i with

$$\alpha_i \log(\frac{\alpha_i \beta_i}{\lambda}) - \frac{\lambda}{\beta_i} (\frac{\alpha_i \beta_i}{\lambda} - 1) = \max_{j \in J} \left\{ \alpha_j \log(\frac{\alpha_j \beta_j}{\lambda}) - \frac{\lambda}{\beta_j} (\frac{\alpha_j \beta_j}{\lambda} - 1) \right\}$$

will be allocated non-zero bandwidth  $W_i$ .

- (d) Develop an algorithm to iteratively compute the optimal value of  $\lambda$ . Describe how to adaptively control  $P_i$  and  $W_i$  based on this iterative algorithm.
- 3. Problems 5.1, 5.21, 5.27, 5.28 in Boyd and Vandenberghe.

**Note:** Please also start working on the rest of the simulation project. (You don't need to submit results with this homework yet.)