# IERG 6120: Homework 4

Due: October 18th, 2023 in class

1. (Scaled gradient descent algorithm) Assume that $f(x)$ is a convex and differentiable function, and its gradient is Lipschitz continuous with modulus $L$. The scaled gradient descent algorithm will follow the iteration:

$$x(t+1) = x(t) - \gamma \Lambda \nabla f(x(t)),$$

where $\Lambda$ is a positive definite matrix. Show that if $0 < \gamma < \frac{2}{L\lambda_{\max}}$, where $\lambda_{\max}$ is the largest eigenvalue of $\Lambda$, then $x(t)$ will converge to a minimizer of $f$.

2. (Verifying smoothness and strong convexity) Consider the following *regularized* linear regression problem:

$$\min_{w} \qquad ||Aw - y||_2^2 + \lambda ||w||_2^2,$$

where $A = [a_1^T, a_2^T, ..., a_n^T]$ denote the independent variables, $y = [y_1, y_2, ..., y_n]^T$ denote the dependent variables, and our goal is to estimate the weights $w$ so that $y_i \approx a_i^T w$ for all $i = 1, 2, ..., n$. The second term with $\lambda > 0$ is called a *regularization term*, which is meant to avoid solutions with $w$ being too large. (When $w$ is very large, the learned model is known to "overfit" the training data, and thus has problems when applied to new test data.)

You are given the following facts. (i) The eigenvalues of the symmetric matrix $A^T A$ are always non-negative. (ii) Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalues of $A^T A$, respectively. Then, for any vector $v$, we must have

$$\lambda_{\min} ||v||_2 \leq ||A^T A v||_2 \leq \lambda_{\max} ||v||_2.$$

   (a) Show that the objective function is always smooth. Provide an upper bound on the smoothness parameter $L$.

   (b) Show that the objective function may not be strongly convex when $\lambda = 0$. Provide a lower bound on the strongly convexity $\alpha$ when $\lambda > 0$. (This is the reason why adding the regularization term $\lambda ||w||_2^2$ helps to also expedite convergence.)

**Note:** Please also start working on Problem 1 of the simulation project. (You don't need to submit results with this homework yet.)