IERG6120: Homework 2

Due: Sept. 25, 2024 in class

Convex Functions

1. A function $||x||$ for $x \in \mathbf{R}^n$ is said to be a *norm* if it satisfies the following properties

   (i) $||x|| \geq 0$ for all $x \in \mathbf{R}^n$.

   (ii) $||x|| = 0$ if and only if $x = 0$.

   (iii) $||\alpha x|| = |\alpha| \, ||x||$ for any real scalar α, where $|\alpha|$ denote the absolute value of α.

   (iv) (Triangle Inequality) $||x + y|| \leq ||x|| + ||y||$.

   Show that any norm is a convex function.

2. Let $f(x)$ be a function of $\mathbf{R} \to \mathbf{R}$ (i.e., it is a function of real numbers). Show that $f(x)$ is convex if and only if $\text{dom} f$ is convex and $f(y) \geq f(x) + f'(x)(y - x)$ for all $x, y \in \text{dom} f$. (Please follow and complete the proof steps described in class.)

3. Are these sets convex? (*Hint:* Write the set in a form $\{x | f(x) \leq 0\}$ where $f(x)$ is a convex function.)

   (a) Let λ be the arrival rate to an $M/M/1$ queueing system with service rate μ, and let d be a delay threshold. The set of $\lambda < \mu$ such that the expected delay $E[D]$ is less than a given value d. What if d is also a variable, i.e., the set of (λ, d), with $\lambda < \mu$, such that the expected delay in the $M/M/1$ queueing system is less than d. (Note: The expected delay $E[D]$ of an $M/M/1$ queue with service rate μ and arrival rate λ is given by $E[D] = 1/(\mu - \lambda)$. For this problem, you can assume that μ is fixed. Alos, we assume $\lambda < \mu$, otherwise the delay expression will be negative.)

   (b) Let P_0 denote the transmission power of a given user. Let P_i, $i = 1, ..., I$ be the interference power of other users. Assume that the transmission rate of user 0 follows the Shannon capacity. The set of vectors $[P_0, P_1, ..., P_I]$ such that the transmission rate of user 0 is greater than a value r. What if r is also a variable and is a component of the vector to be considered. (Note: the Shannon capacity C of a channel with transmission power P, interference power I, and noise power N is given by $C = W \log(1+P/(I+N))$, where W is the bandwidth of the system.

For this problem, you can assume that W and N are fixed. The interference power I corresponds to the total transmission power of users other than user 0. For this problem, please do NOT use the high (or low) SINR approximation.)

(c) Let x and y denote the length and width, respectively, of a rectangle contained in a circle with radius r. The set of such (x, y) for a given r, and the set of such (x, y, r).

4. Problem 3.5, 3.16 (part a/c/e), and 3.21 (part a) of Boyd and Vandenberghe. (You don't need to worry about the quasi-convex and quasi-concave part of the problems.)