The Bethe Partition Function and the SPA for Factor Graphs based on Homogeneous Real Stable Polynomials

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***Abstract*—Various computational problems can be reduced to computing the marginals and the partition function of a suitably defined standard factor graph (S-FG). The sum-product algorithm (SPA) is an efficient iterative method for approximating these quantities, resulting in the so-called Bethe approximation of these quantities.**

In previous work, Vontobel proved that for an S-FG whose partition function equals the permanent of a nonnegative square matrix, the Bethe free energy function associated with the S-FG is a convex function and the SPA efficiently finds the minimum of the Bethe free energy function, from which the Bethe approximation of the permanent can be computed.

We extend Vontobel's results by considering a class of bipartite S-FGs where each local function is defined based on a (possibly different) multi-affine homogeneous real stable polynomial. This class of S-FGs covers various combinatorial problems, including computing a generalization of the matrix permanent and determining the number of binary contingency tables with prescribed marginals. Results by Straszak and Vishnoi for a slightly larger class of S-FGs (they do not assume homogeneity of the polynomials) show that these S-FGs have the property that the Bethe partition function lower bounds the partition function.

In this paper we prove, with the help of results for real stable polynomials and results from matroid theory, various statements for the class of S-FGs under consideration: we show that a certain projection of the local marginal polytope equals the convex hull of the set of valid configurations, that the Bethe free energy function possesses some convexity properties, and, for the typical case where the S-FG has an SPA fixed point consisting of positive-valued messages only, that the SPA finds the value of the Bethe partition function exponentially fast.

I. Introduction

Several fundamental computational problems in statistical mechanics, coding theory, and computer science can be formulated as computational tasks regarding multivariate functions. A typical task is computing the marginals and the sum of a multivariate nonnegative real-valued function that is given by the product of nonnegative real-valued functions [1]. The factorization of this function can be visualized by a standard factor graph (S-FG), and the marginals and the sum of this function are related to computing the marginals and partition function of the S-FG, respectively [2]–[4].

The sum-product algorithm (SPA), also known as loopy belief propagation (LBP), is one of the most widely deployed methods to approximate the marginals and the partition function of an S-FG [3]–[5]. If the S-FG is cycle-free, then the SPA gives the exact marginals and the exact partition function at its only fixed point. If the S-FG has cycles, then the SPA often provides a decent approximation to these quantities.

Yedidia *et al.* [6] connected the SPA fixed points to the stationary points of the Bethe free energy function associated with an S-FG. They further introduced the Bethe approximation to the partition function of the S-FG, the so-called Bethe partition function, which is defined in terms of the minimum of the Bethe free energy function. For many S-FGs of interest, the Bethe partition function gives a surprisingly good estimate to the partition function [7]. Nevertheless, there are known example S-FGs for which the SPA fails to converge or for which the Bethe partition function is a poor approximation to the partition function [8]–[10].

There are S-FGs where the SPA exhibits favorable behavior, and the Bethe partition function provides a reasonable estimate of the partition function. For the S-FG whose partition function equals the permanent of a nonnegative square matrix, Vontobel in [11] showed that the Bethe free energy function is a convex function and that the SPA finds its minimum exponentially fast. Furthermore, Gurvits in [12] proved that the Bethe partition function lower bounds the permanent, and Anari and Razaei [13] proved that the permanent is upper bounded by $2^{n/2}$ times the Bethe partition function, where n is the number of the rows and columns of the matrix. Afterwards, Huang *et al.* in [14] gave a combinatorial characterization of the above-mentioned bounds for the matrix permanent via the degree-M Bethe partition function.

A natural question is whether we can generalize the aforementioned results, *i.e.*, whether we can find classes of S-FGs such that the SPA and the Bethe partition function are still well-behaved. In this paper, motivated by recent developments in the area of polynomial approaches to approximating partition functions [15], [16] and in the theory of real stable polynomials [17]–[19], we consider bipartite S-FGs where each local function is defined based on a (possibly different) multi-affine homogeneous real stable (MAHRS) polynomial. Such S-FGs can be viewed as a generalization of the S-FGs for the matrix permanent in [11]: by suitably defining the local functions in such an S-FG, computing the associated partition function is equivalent to solving various fundamental combinatorial problems. Straszak and Vishnoi [20] proved that for S-FGs where each local function corresponds to a multi-affine real stable polynomial (note that they do not assume homogeneity), the Bethe partition function lower bounds the partition function.

In this paper, for an S-FG from the class of S-FGs under consideration, we investigate the properties of its Bethe partition function, along with studying the behavior of the SPA. The main contributions of this paper are as follows.

1) We show that by suitably defining the local functions of the S-FG, computing the associated partition function is equivalent to solving various fundamental combinatorial problems, *e.g.*, finding the number of binary contingency tables with prescribed marginals [21].
2) We prove that a certain projection of the local marginal polytope of the S-FG equals the convex hull of the set of valid configurations of the S-FG. We do this, first, by using results about MAHRS polynomials in [22] that show that the support of each local function in the S-FG is the set of bases of a suitably defined matroid, and, second, by using the celebrated matroid intersection results in [23].
3) We present primal and dual formulations of the Bethe partition function. In particular, we show that the pri-

mal formulation corresponds to a convex optimization problem and that the dual formulation corresponds to a convex-concave minimax optimization problem. Our proposed dual formulation is different from the formulation of the Bethe partition function proposed in [20].

4) We prove, with the help of results about MAHRS polynomials in [22], that the SPA converges exponentially fast to the fixed point, and that the value of the Bethe partition function can be evaluated by the fixed-point messages. Note that, due to space constraints, we prove this result only for the, typically occurring, case where the S-FG has an SPA fixed point consisting of positive-valued messages only.

Note that the proofs of most of these results require non-trivial generalizations of the proofs of the results in [11].

The rest of this paper is structured as follows. Section II formally defines the considered S-FGs. Section III introduces the primal and dual formulations of the Bethe partition function. Section IV studies the SPA on the considered S-FGs. The appendices provide the proofs of the main results in this paper.

A. Basic Notations and Definitions

All logarithms are natural logarithms. As usually done in information theory, we define $\log(0) \triangleq -\infty$, $0 \cdot \log 0 \triangleq 0$, and $0^0 \triangleq 1$. For any positive integer $n \in \mathbb{Z}_{\geq 1}$, we define $[n] \triangleq \{1, \ldots, n\}$. For any length-$n$ vector $\boldsymbol{x} = \bigl(x(1), \ldots, x(n)\bigr) \in \{0,1\}^n$, we define $w_{\mathrm{H}}(\boldsymbol{x})$ to be the Hamming weight of the vector $\boldsymbol{x}$, *i.e.*, $w_{\mathrm{H}}(\boldsymbol{x}) \triangleq \sum_{i=1}^{n} x(i)$.

Definition 1. *For any finite set $\mathcal{S}$ and any collections of variables $\boldsymbol{A} \triangleq \bigl(A(s)\bigr)_{s \in \mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$ and $\boldsymbol{B} \triangleq \bigl(B(s)\bigr)_{s \in \mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$, we define*

$$\boldsymbol{A}^{\boldsymbol{B}} \triangleq \prod_{s \in \mathcal{S}} \bigl(A(s)\bigr)^{B(s)}, \qquad \langle \boldsymbol{A}, \boldsymbol{B} \rangle \triangleq \sum_{s \in \mathcal{S}} A(s) \cdot B(s),$$

$$\boldsymbol{A} \cdot \boldsymbol{B} \triangleq \bigl(A(s) \cdot B(s)\bigr)_{s \in \mathcal{S}}, \quad \exp(\boldsymbol{A}) \triangleq \Bigl(\exp\bigl(A(s)\bigr)\Bigr)_{s \in \mathcal{S}},$$

$$\frac{\boldsymbol{A}}{\boldsymbol{B}} \triangleq \left(\frac{A(s)}{B(s)}\right)_{s \in \mathcal{S}}, \quad B(s) \neq 0, \, \forall s \in \mathcal{S}.$$

Definition 2. *Consider a finite set $\mathcal{S}$. Define the set*

$$\Pi_{\mathcal{S}} \triangleq \left\{ \boldsymbol{p} \triangleq \bigl(p(s)\bigr)_{s \in \mathcal{S}} \;\middle|\; \sum_{s \in \mathcal{S}} p(s) = 1, \, p(s) \geq 0, \, \forall s \in \mathcal{S} \right\}$$

to be the set of probability mass functions (PMFs) over $\mathcal{S}$. Consider $n \in \mathbb{Z}_{\geq 1}$ and a collection of vectors $\{\boldsymbol{x}_s\}_{s \in \mathcal{S}}$ such that $\boldsymbol{x}_s \in \mathbb{R}^n$ for all $s \in \mathcal{S}$. Define the convex hull of $\{\boldsymbol{x}_s\}_{s \in \mathcal{S}}$ to be

$$\mathrm{conv}(\{\boldsymbol{x}_s\}_{s \in \mathcal{S}}) \triangleq \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \exists \boldsymbol{p} \in \Pi_{\mathcal{S}} \text{ s.t. } \boldsymbol{x} = \sum_{s \in \mathcal{S}} p(s) \cdot \boldsymbol{x}_s \right\}.$$

II. An MAHRS Polynomials-based Standard Normal Factor Graph (S-NFG)

In this section, we introduce a standard normal factor graph (S-NFG)[1] where each local function corresponds to an MAHRS polynomial. (For the definition and properties of MAHRS polynomials, see Appendix A.) We will see that many fundamental combinatorial problems are related to exactly computing the partition function of such an S-NFG.

Definition 3. *Fix some $n, m \in \mathbb{Z}_{\geq 1}$. Define $\mathcal{I} = [n]$ and $\mathcal{J} = [m]$. We consider a bipartite S-NFG $\mathsf{N}(\mathcal{F}, \mathcal{E}, \mathcal{X})$ consisting of the following objects:*

[1]In this paper, we will consider, without essential loss of generality, S-NFGs, *i.e.*, S-FGs where variables are associated with edges [2], [3].

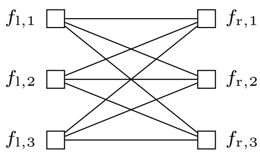

Fig. 1: An example S-NFG N with $n = m = 3$.

1) *The graph $(\mathcal{F}, \mathcal{E})$, where the set of function nodes $\mathcal{F}$ is defined to be $\mathcal{F} \triangleq \{f_{\mathrm{l},i}\}_{i \in \mathcal{I}} \cup \{f_{\mathrm{r},j}\}_{j \in \mathcal{J}}$ with $f_{\mathrm{l},i}$ representing the i-th function node on the left-hand side (LHS) and $f_{\mathrm{r},j}$ representing the j-th function node on the right-hand side (RHS), and the set of edges is defined to be $\mathcal{E} \subseteq \mathcal{I} \times \mathcal{J}$. If $(i,j) \in \mathcal{E}$, there is an edge connecting function nodes $f_{\mathrm{l},i}$ and $f_{\mathrm{r},j}$ in N. The set $\mathcal{F}$ is also known as the set of local functions.*
2) *The alphabet $\mathcal{X} \triangleq \prod_{(i,j) \in \mathcal{E}} \mathcal{X}_{i,j}$, where the alphabet associated with the edge $(i,j) \in \mathcal{E}$ is $\mathcal{X}_{i,j} \triangleq \{0,1\}$ and the corresponding variable is denoted by $x(i,j)$.*

An example S-NFG with $n = m = 3$ is shown in Fig. 1. We make some further definitions for N as follows.

3) *Define $\boldsymbol{x} \triangleq \bigl(x(i,j)\bigr)_{(i,j) \in \mathcal{E}} \in \mathcal{X}$ to be a configuration of N.*
4) *For each $i \in \mathcal{I}$, define*
$$\mathcal{J}_i \triangleq \{ j' \in [\mathcal{J}] \mid (i,j') \text{ is incident on } f_{\mathrm{l},i} \}$$
and $m_i \triangleq |\mathcal{J}_i|$.
5) *For each $j \in \mathcal{J}$, define*
$$\mathcal{I}_j \triangleq \{ i' \in [\mathcal{I}] \mid (i',j) \text{ is incident on } f_{\mathrm{r},j} \}$$
and $n_j \triangleq |\mathcal{I}_j|$.
6) *For any $\boldsymbol{A} \triangleq \bigl(A(i,j)\bigr)_{(i,j) \in \mathcal{E}} \in \mathbb{C}^{|\mathcal{E}|}$, define the vectors $\boldsymbol{A}(i,:) \triangleq \bigl(A(i,j')\bigr)_{j' \in \mathcal{J}_i}$ and $\boldsymbol{A}(:,j) \triangleq \bigl(A(i',j)\bigr)_{i' \in \mathcal{I}_j}$.*
7) *Define $\boldsymbol{L} \triangleq \bigl(L(i,j)\bigr)_{(i,j) \in \mathcal{E}}$ and $\boldsymbol{R} \triangleq \bigl(R(i,j)\bigr)_{(i,j) \in \mathcal{E}}$ to be collections of variables that take value in $\mathbb{C}^{|\mathcal{E}|}$.*
8) *For each $i \in \mathcal{I}$, define the local function $f_{\mathrm{l},i}$ to be an arbitrary mapping from $\prod_{j \in \mathcal{J}_i} \mathcal{X}_{i,j}$ to $\mathbb{R}_{\geq 0}$ such that*
$$p_i\bigl(\boldsymbol{L}(i,:)\bigr) \triangleq \sum_{\boldsymbol{x}(i,:) \in \{0,1\}^{m_i}} f_{\mathrm{l},i}\bigl(\boldsymbol{x}(i,:)\bigr) \cdot \bigl(\boldsymbol{L}(i,:)\bigr)^{\boldsymbol{x}(i,:)}$$
is an MAHRS polynomial of degree r_i with respect to the variables in $\boldsymbol{L}(i,:)$. Let $\mathcal{X}_{f_{\mathrm{l},i}}$ be the support of $f_{\mathrm{l},i}$, i.e.,
$$\mathcal{X}_{f_{\mathrm{l},i}} \triangleq \bigl\{ \boldsymbol{x}(i,:) \in \{0,1\}^{m_i} \bigm| f_{\mathrm{l},i}\bigl(\boldsymbol{x}(i,:)\bigr) > 0 \bigr\}.$$
9) *For each $j \in \mathcal{J}$, define the local function $f_{\mathrm{r},j}$ to be an arbitrary mapping from $\prod_{i \in \mathcal{I}_j} \mathcal{X}_{i,j}$ to $\mathbb{R}_{\geq 0}$ such that*
$$q_j\bigl(\boldsymbol{R}(:,j)\bigr) \triangleq \sum_{\boldsymbol{x}(:,j) \in \{0,1\}^{n_j}} f_{\mathrm{r},j}\bigl(\boldsymbol{x}(:,j)\bigr) \cdot \bigl(\boldsymbol{R}(:,j)\bigr)^{\boldsymbol{x}(:,j)}$$
is an MAHRS polynomial of degree c_j with respect to the variables in $\boldsymbol{R}(:,j)$. Let $\mathcal{X}_{f_{\mathrm{r},j}}$ be the support of $f_{\mathrm{r},j}$, i.e.,
$$\mathcal{X}_{f_{\mathrm{r},j}} \triangleq \bigl\{ \boldsymbol{x}(:,j) \in \{0,1\}^{n_j} \bigm| f_{\mathrm{r},j}\bigl(\boldsymbol{x}(:,j)\bigr) > 0 \bigr\}.$$
10) *For any $\boldsymbol{x} \in \mathcal{X}$, define the global function g to be*
$$g(\boldsymbol{x}) \triangleq \left(\prod_{i \in \mathcal{I}} f_{\mathrm{l},i}\bigl(\boldsymbol{x}(i,:)\bigr) \right) \cdot \left(\prod_{j \in \mathcal{J}} f_{\mathrm{r},j}\bigl(\boldsymbol{x}(:,j)\bigr) \right).$$
11) *Define the set $\mathcal{C}$ to be the set of valid configurations:*
$$\mathcal{C} \triangleq \{ \boldsymbol{x} \in \mathcal{X} \mid g(\boldsymbol{x}) > 0 \}.$$
12) *Define the partition function to be $Z(\mathsf{N}) \triangleq \sum_{\boldsymbol{x} \in \mathcal{C}} g(\boldsymbol{x})$.*

Assumption 4. *We assume that $Z(\mathsf{N}) > 0$.*

For convenience, if there is no ambiguity, we use shorthands $\sum_{\boldsymbol{x}(i,:)}$, $\sum_{\boldsymbol{x}(:,j)}$, $\sum_{x(i,j)}$, $\sum_{(i,j)}$, $\sum_{i}$, $\sum_{j}$, $\prod_{\boldsymbol{x}(i,:)}$, $\prod_{\boldsymbol{x}(:,j)}$, $\prod_{x(i,j)}$, $\prod_{(i,j)}$, $\prod_{i}$, $\prod_{j}$, $\prod_{\boldsymbol{x}}$, $\{\ \}_{(i,j)}$, $(\ )_{i}$, $(\ )_{j}$, $(\ )_{(i,j)}$, $(\ )_{\boldsymbol{x}(i,:)}$, $(\ )_{\boldsymbol{x}(:,j)}$, and $(\ )_{\boldsymbol{x}}$, for $\sum_{\boldsymbol{x}(i,:)\in\mathcal{X}_{f_{1,i}}}$ $\sum_{\boldsymbol{x}(:,j)\in\mathcal{X}_{f_{\mathrm{r},j}}}$ $\sum_{x(i,j)\in\mathcal{X}_{i,j}}$, $\sum_{(i,j)\in\mathcal{E}}$, $\sum_{i\in\mathcal{I}}$, $\sum_{j\in\mathcal{J}}$, $\prod_{\boldsymbol{x}(i,:)\in\mathcal{X}_{f_{1,i}}}$ $\prod_{\boldsymbol{x}(:,j)\in\mathcal{X}_{f_{\mathrm{r},j}}}$ $\prod_{x(i,j)\in\mathcal{X}_{i,j}}$, $\prod_{(i,j)\in\mathcal{E}}$, $\prod_{i\in\mathcal{I}}$, $\prod_{j\in\mathcal{J}}$, $\prod_{\boldsymbol{x}\in\mathcal{X}}$, $\{\ \}_{(i,j)\in\mathcal{E}}$, $(\ )_{i\in\mathcal{I}}$, $(\ )_{j\in\mathcal{J}}$, $(\ )_{(i,j)\in\mathcal{E}}$, $(\ )_{\boldsymbol{x}(i,:)\in\mathcal{X}_{f_{1,i}}}$, $(\ )_{\boldsymbol{x}(:,j)\in\mathcal{X}_{f_{\mathrm{r},j}}}$, and $(\ )_{\boldsymbol{x}\in\mathcal{X}}$, respectively. For convenience, for any collection of variables $\boldsymbol{A} \triangleq (A(i,j))_{(i,j)}$ and any integers $i' \in \mathcal{I}$ and $j' \in \mathcal{J}$, we use the shorthands $\sum_j A(i',j)$ and $\sum_i A(i,j')$ for $\sum_{j\in\mathcal{J}_{i'}} A(i',j)$ and $\sum_{i\in\mathcal{I}_{j'}} A(i,j')$, respectively, if there is no ambiguity.

In following example, we relate the exact computation of the partition function for the class of the S-NFGs defined in Definition 3 to various fundamental combinatorial problems. In particular, we show that Definition 3 generalizes the definition of the S-NFG in [11, Definition 4], whose partition function equals the permanent of a nonnegative square matrix.

Example 5. *Consider an arbitrary positive-valued matrix of size $n \times m$: $\boldsymbol{\theta} \triangleq \big(\theta(i,j)\big)_{(i,j)} \in \mathbb{R}^{n\times m}_{>0}$. Let $\mathcal{J}_i = [m]$ and $\mathcal{I}_j = [n]$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Fix $r_i \in [m]$ and $c_j \in [n]$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$. For each $i \in \mathcal{I}$ and $j \in \mathcal{J}$, the local functions are defined to be*

$$f_{1,i}\big(\boldsymbol{x}(i,:)\big) \triangleq \begin{cases} \big(\boldsymbol{\theta}(i,:)\big)^{\boldsymbol{x}(i,:)/2} & \text{if } w_{\mathrm{H}}\big(\boldsymbol{x}(i,:)\big) = r_i \\ 0 & \text{otherwise} \end{cases},$$

$$f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \triangleq \begin{cases} \big(\boldsymbol{\theta}(:,j)\big)^{\boldsymbol{x}(:,j)/2} & \text{if } w_{\mathrm{H}}\big(\boldsymbol{x}(:,j)\big) = c_j \\ 0 & \text{otherwise} \end{cases}.$$

This example S-NFG has the following properties.

1) *For each $i \in [n]$, the polynomial $p_i\big(\boldsymbol{L}(i,:)\big)$ is an elementary symmetric polynomial of degree r_i, i.e., an MAHRS polynomial. The set $\mathcal{X}_{f_{1,i}}$ corresponds to a uniform matroid of rank r_i. Similarly, for each $j \in [m]$, the polynomial $q_j\big(\boldsymbol{R}(:,j)\big)$ is an elementary symmetric polynomial of degree c_j, i.e., an MAHRS polynomial, and the set $\mathcal{X}_{f_{\mathrm{r},j}}$ corresponds to a uniform matroid of rank c_j. (For details, see the discussion in Appendices B and C, which are recommended to be read after reading the proof of Theorem 8.)*
2) *It holds that*

$$\mathcal{C} = \left\{ \boldsymbol{x} \in \mathcal{X} \;\middle|\; \begin{array}{l} w_{\mathrm{H}}\big(\boldsymbol{x}(i,:)\big) = r_i,\ \forall i \in \mathcal{I} \\ w_{\mathrm{H}}\big(\boldsymbol{x}(:,j)\big) = c_j,\ \forall j \in \mathcal{J} \end{array} \right\}.$$

   By [24, Section 6.2], we know that the set $\mathcal{C}$ is non-empty if and only if $\sum_{i\in\mathcal{I}} r_i = \sum_{j\in\mathcal{J}} c_j$ and for all subsets $\mathcal{S} \subseteq [m]$, we have $\sum_{j\in\mathcal{S}} c_j \leq \sum_{i\in\mathcal{I}} \min\{r_i, |\mathcal{S}|\}$.
3) *If $n = m$ and $r_i = c_j = 1$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$, then $Z(\mathsf{N})$ equals the permanent of $\boldsymbol{\theta}$ (see, e.g., [25]). Computing $Z(\mathsf{N})$ in this case is in the complexity class #P, where #P is the set of the counting problems that correspond to the decision problems in the class NP. Remarkably, even if we consider $\boldsymbol{\theta} \in \{0,1\}^{n\times m}$ instead of $\boldsymbol{\theta} \in \mathbb{R}^{n\times m}_{>0}$, the computation of $Z(\mathsf{N})$ is #P-complete [26].*
4) *The partition function $Z(\mathsf{N}) = \sum_{\boldsymbol{x}\in\mathcal{C}} \boldsymbol{\theta}^{\boldsymbol{x}}$ can be viewed as a generalization of the permanent of $\boldsymbol{\theta}$. If $\boldsymbol{\theta}$ is a matrix of ones, then $Z(\mathsf{N})$ equals the number of binary matrices with prescribed row and column sums, i.e., the number of binary contingency tables with prescribed marginals. (See, e.g., [21], [27], [28].)*

III. The Primal and Dual Formulations of the Bethe Partition Function

In the previous section, we showed that computing the partition function for the considered S-NFG is in general non-trivial. In this section, we introduce the primal and dual formulations of the Bethe partition function, which provides a graphical-model-based approach to approximate the partition function. We present convexity and concavity properties of these formulations. Before giving the definition of these formulations, we need to introduce the local marginal polytope. (For the details of local marginal polytope, see, *e.g.*, [7, Section 4.1.1].)

Definition 6. *We make the following definitions.*

1) *Define a collection of beliefs: $\boldsymbol{\beta} \triangleq \big((\boldsymbol{\beta}_i)_i, (\boldsymbol{\beta}_j)_j, \boldsymbol{\beta}_{\mathcal{E}}\big)$, with $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$ corresponding to the beliefs of the function nodes $f_{1,i}$ and $f_{\mathrm{r},j}$, respectively, and $\boldsymbol{\beta}_{\mathcal{E}}$ corresponding to the beliefs of the edges:*[2]

$$\boldsymbol{\beta}_i \triangleq \big(\beta_i(\boldsymbol{x}(i,:))\big)_{\boldsymbol{x}(i,:)}, \quad \boldsymbol{\beta}_j \triangleq \big(\beta_j(\boldsymbol{x}(:,j))\big)_{\boldsymbol{x}(:,j)},$$
$$\boldsymbol{\beta}_{\mathcal{E}} \triangleq \big(\beta_{\mathcal{E}}(i,j)\big)_{(i,j)}.$$

2) *Define $\boldsymbol{\beta}_{\mathcal{I}} \triangleq \big(\beta_{\mathcal{I}}(i,j)\big)_{(i,j)}$ and $\boldsymbol{\beta}_{\mathcal{J}} \triangleq \big(\beta_{\mathcal{J}}(i,j)\big)_{(i,j)}$ to be the collections of the marginals of the beliefs $(\boldsymbol{\beta}_i)_i$ and $(\boldsymbol{\beta}_j)_j$, where*

$$\beta_{\mathcal{I}}(i,j) \triangleq \sum_{\boldsymbol{x}(i,:):\, x(i,j)=1} \beta_i\big(\boldsymbol{x}(i,:)\big),$$
$$\beta_{\mathcal{J}}(i,j) \triangleq \sum_{\boldsymbol{x}(:,j):\, x(i,j)=1} \beta_j\big(\boldsymbol{x}(:,j)\big).$$

3) *The local marginal polytope (LMP) is defined to be the set*

$$\mathcal{L} \triangleq \left\{ \boldsymbol{\beta} \;\middle|\; \begin{array}{c} \boldsymbol{\beta}_i \in \Pi_{\mathcal{X}_{f_{1,i}}},\ \forall i \in \mathcal{I} \\ \boldsymbol{\beta}_j \in \Pi_{\mathcal{X}_{f_{\mathrm{r},j}}},\ \forall j \in \mathcal{J} \\ \boldsymbol{\beta}_{\mathcal{I}} = \boldsymbol{\beta}_{\mathcal{J}} = \boldsymbol{\beta}_{\mathcal{E}} \end{array} \right\}.$$

4) *The polytope $\mathcal{L}_{\mathcal{E}}$ is defined to be the projection of the LMP $\mathcal{L}$ onto the beliefs of the edges, i.e.,*

$$\mathcal{L}_{\mathcal{E}} \triangleq \left\{ \boldsymbol{\beta}_{\mathcal{E}} \;\middle|\; \begin{array}{l} \boldsymbol{\beta}_{\mathcal{E}}(i,:) \in \mathrm{conv}(\mathcal{X}_{f_{1,i}}),\ \forall i \in \mathcal{I} \\ \boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathrm{conv}(\mathcal{X}_{f_{\mathrm{r},j}}),\ \forall j \in \mathcal{J} \end{array} \right\}.$$

Proposition 7. *For any $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$, there exists a $\boldsymbol{\beta}' \in \mathcal{L}$ such that $\boldsymbol{\beta}_{\mathcal{E}} = \boldsymbol{\beta}'_{\mathcal{E}}$.*

Proof. This follows straightforwardly from the definitions of $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{L}$ in Definition 6. ■

Theorem 8. *The convex hull of all the valid configurations of* N *is equal to the set $\mathcal{L}_{\mathcal{E}}$, i.e.,* $\mathrm{conv}(\mathcal{C}) = \mathcal{L}_{\mathcal{E}}$.

Proof. The proof is mainly relies on results from matroid theory. For details, see Appendix B. ■

If we consider the example S-NFG in Item 3 of Example 5, then Vontobel in [11, Lemma 13] showed that in this case, Theorem 8 is equivalent to the Birkhoff-von Neumann theorem. (For the Birkhoff–von Neumann theorem, see, *e.g.*, [29, Theorem 4.3.49].)

Definition 9. *The Bethe free energy function associated with* N *is defined to be the function $F_{\mathrm{B}}(\boldsymbol{\beta}) \triangleq U_{\mathrm{B}}(\boldsymbol{\beta}) - H_{\mathrm{B}}(\boldsymbol{\beta})$, where $\boldsymbol{\beta} \in \mathcal{L}$ and*

$$U_{\mathrm{B}}(\boldsymbol{\beta}) \triangleq -\sum_{i} \sum_{\boldsymbol{x}(i,:)} \beta_i\big(\boldsymbol{x}(i,:)\big) \cdot \log\Big(f_{1,i}\big(\boldsymbol{x}(i,:)\big)\Big)$$

[2]For each $(i,j) \in \mathcal{E}$, the belief $\beta_{\mathcal{E}}(i,j)$ corresponds to the belief that $x(i,j) = 1$, and $1 - \beta_{\mathcal{E}}(i,j)$ corresponds to the belief that $x(i,j) = 0$.

$$-\sum_j \sum_{\boldsymbol{x}(:,j)} \beta_j\big(\boldsymbol{x}(:,j)\big) \cdot \log\Big(f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big)\Big),$$

$$\begin{aligned} H_{\mathrm{B}}(\boldsymbol{\beta}) \triangleq &-\sum_i \sum_{\boldsymbol{x}(i,:)} \beta_i\big(\boldsymbol{x}(i,:)\big) \cdot \log\Big(\beta_i\big(\boldsymbol{x}(i,:)\big)\Big) \\ &-\sum_j \sum_{\boldsymbol{x}(:,j)} \beta_j\big(\boldsymbol{x}(:,j)\big) \cdot \log\Big(\beta_j\big(\boldsymbol{x}(:,j)\big)\Big) \\ &+\sum_{(i,j)} \beta_{\mathcal{E}}(i,j) \cdot \log\big(\beta_{\mathcal{E}}(i,j)\big) \\ &+\sum_{(i,j)} \big(1-\beta_{\mathcal{E}}(i,j)\big) \cdot \log\big(1-\beta_{\mathcal{E}}(i,j)\big). \end{aligned}$$

Definition 10. *The Bethe approximation to the partition function is defined to be* $Z_{\mathrm{B}}(\mathsf{N}) \triangleq \exp\big(-\min_{\boldsymbol{\beta}\in\mathcal{L}(\mathsf{N})} F_{\mathrm{B}}(\boldsymbol{\beta})\big)$. *In the following, we call this the primal formulation of Bethe partition function.*

A natural question is whether we can bound the partition function via the Bethe partition function. For the S-NFG in Item 3 of Example 5, Gurvits in [12, Theorem 2.2] proved that $Z(\mathsf{N}) \geq Z_{\mathrm{B}}(\mathsf{N})$. The following theorem generalizes the lower bound proven by Gurvits.

Theorem 11. *([20, Theorem 3.2]; see also [15, Theorem 1.1]) For the S-NFG defined in Definition 3, it holds that*

$$Z(\mathsf{N}) \geq Z_{\mathrm{B}}(\mathsf{N}).$$

Actually, Straszak and Vishnoi in [20, Theorem 3.2] proved that the Bethe partition function lower bounds the partition function even for the S-NFGs where each local function corresponds to a multi-affine real stable polynomial, *i.e.*, they do not assume homogeneity.

In the following, we give an modified version of the primal formulation in Definition 10 and also a dual formation of the Bethe partition function. By studying the convexity and concavity of the objective function in the dual formulation, we can relate the modified version of the primal formulation to a convex optimization problem.

Motivated by Proposition 7, we first define a convex set based on the beliefs of the edges.

Definition 12. *For any* $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$, *we define the set of the beliefs of the function nodes that realize* $\boldsymbol{\beta}_{\mathcal{E}}$, *to be the following set:*

$$\mathcal{L}_{\mathrm{margin}}\big(\boldsymbol{\beta}_{\mathcal{E}}\big) \triangleq \left\{ \big((\boldsymbol{\beta}_i)_i, (\boldsymbol{\beta}_j)_j\big) \;\middle|\; \begin{array}{c} \boldsymbol{\beta}_{\mathcal{I}} = \boldsymbol{\beta}_{\mathcal{J}} = \boldsymbol{\beta}_{\mathcal{E}} \\ \boldsymbol{\beta}_i \in \Pi_{\mathcal{X}_{f_{\mathrm{l},i}}}, \forall i \in \mathcal{I}, \\ \boldsymbol{\beta}_j \in \Pi_{\mathcal{X}_{f_{\mathrm{r},j}}}, \forall j \in \mathcal{J} \end{array} \right\}.$$

We also define

$$F_{\mathrm{B},\mathcal{E}}(\boldsymbol{\beta}_{\mathcal{E}}) \triangleq \min_{((\boldsymbol{\beta}_i)_i,\, (\boldsymbol{\beta}_j)_j) \in \mathcal{L}_{\mathrm{margin}}(\boldsymbol{\beta}_{\mathcal{E}})} F_{\mathrm{B}}(\boldsymbol{\beta}).$$

Based on the primal formulation of the Bethe partition function in Definition 10, we obtain a modified version of the primal formulation based on $F_{\mathrm{B},\mathcal{E}}$ as follows:

$$Z_{\mathrm{B}}(\mathsf{N}) = \exp\Big(-\min_{\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}} F_{\mathrm{B},\mathcal{E}}(\boldsymbol{\beta}_{\mathcal{E}})\Big). \tag{1}$$

Now we introduce a dual formulation of the Bethe partition function and show how it corresponds to a convex-concave minimax optimization problem. The intuition behind the dual formulation is as follows. The primal formulation in Definition 10 is given by the minimum of the Bethe free energy function, which is related to a sum of entropy functions. Following a similar idea in the derivation of the result that the conjugate dual of the entropy function is a "log-sum-exponential" function (see, *e.g.*, [30, Example 3.25]), we apply the Lagrangian dual method to the primal formulation and get a dual formulation that is related to a sum of "log-sum-exponential" functions.

Definition 13. *A minimax problem is defined as follows.*

1) *For each* $i \in \mathcal{I}$ *and* $j \in \mathcal{J}$, *define*

$$\begin{aligned} \tilde{p}_i\big(\boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:)\big) &\triangleq \sum_{\boldsymbol{x}(i,:)} f_{\mathrm{l},i}\big(\boldsymbol{x}(i,:)\big) \cdot \mathrm{e}^{\langle \boldsymbol{L}(i,:), \boldsymbol{x}(i,:)\rangle} \\ &\qquad \cdot \big(\boldsymbol{\beta}_{\mathcal{E}}(i,:)\big)^{\boldsymbol{x}(i,:)}, \\ \tilde{q}_j\big(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big) &\triangleq \sum_{\boldsymbol{x}(:,j)} f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \cdot \mathrm{e}^{\langle \boldsymbol{R}(:,j), \boldsymbol{x}(:,j)\rangle} \\ &\qquad \cdot \big(\boldsymbol{1} - \boldsymbol{\beta}_{\mathcal{E}}(:,j)\big)^{\boldsymbol{1}-\boldsymbol{x}(:,j)}. \end{aligned}$$

2) *Define*

$$\tilde{Z}_{\mathrm{B,d}}(\mathsf{N}) \triangleq \exp\bigg(-\min_{\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}} \sup_{\boldsymbol{L},\boldsymbol{R} \in \mathbb{R}^{|\mathcal{E}|}} \tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}, \boldsymbol{R}\big)\bigg),$$

where

$$\begin{aligned} \tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}, \boldsymbol{R}\big) \triangleq &-\sum_i \log\Big(\tilde{p}_i\big(\boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:)\big)\Big) \\ &-\sum_j \log\Big(\tilde{q}_j\big(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big)\Big) + \langle \boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}+\boldsymbol{R}\rangle. \end{aligned}$$

Theorem 14. *The definition of* $\tilde{Z}_{\mathrm{B,d}}(\mathsf{N})$ *in Definition 13 provides a dual formulation of* $Z_{\mathrm{B}}(\mathsf{N})$, *i.e.,* $\tilde{Z}_{\mathrm{B,d}}(\mathsf{N}) = Z_{\mathrm{B}}(\mathsf{N})$.

Proof. See Appendix D. ∎

Theorem 15. *The following properties of* $\tilde{F}_{\mathrm{B,d}}$ *hold.*

1) *The function* $\tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}, \boldsymbol{R}\big)$ *is a convex function with respect to* $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$ *for fixed* $\boldsymbol{L}, \boldsymbol{R} \in \mathbb{R}^{|\mathcal{E}|}$.
2) *The function* $\tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}, \boldsymbol{R}\big)$ *is a concave function with respect to* $\boldsymbol{L}, \boldsymbol{R} \in \mathbb{R}^{|\mathcal{E}|}$ *for fixed* $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$.

Because both the set $\mathcal{L}_{\mathcal{E}}$ *and the set* $\mathbb{R}^{|\mathcal{E}|}$ *are convex, we further obtain that* $\tilde{Z}_{\mathrm{B,d}}(\mathsf{N})$ *corresponds to a convex-concave minimax optimization problem.*

Proof. See Appendix E. ∎

Corollary 16. *The function* $F_{\mathrm{B},\mathcal{E}}$ *is a convex function with respect to* $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$, *and the minimization problem in the modified version* (1) *of the primal formulation of* $Z_{\mathrm{B}}(\mathsf{N})$ *is a convex optimization problem.*

Proof. See Appendix F. ∎

If we consider the example S-NFG in Item 3 of Example 5, then Corollary 16 is equivalent to [11, Corollary 23].

IV. Behavior of the Sum-Product Algorithm

Given that both the primal and the dual formulations of the Bethe partition function have good properties for the considered class of S-NFGs, we expect the sum-product algorithm (SPA) to behave well for this class of S-NFGs. In this section, due the space constraints, we focus on a typical case where the S-NFG has an SPA fixed point that consists of positive-valued messages only. We show that in this case the SPA converges exponentially fast to this SPA fixed point and that the value of the Bethe partition function can be evaluated by the associated SPA fixed-point messages.

Definition 17. *We make some basic definitions for the SPA on the S-NFG* N *as follows.*

- *Define* $t \in \mathbb{Z}_{\geq 0}$ *to be the iteration index.*
- *For every* $t \in \mathbb{Z}_{\geq 0}$ *and* $(i,j) \in \mathcal{E}$, *define a left-going message* $\overleftarrow{\mu}_{t,i,j} : \mathcal{X}_{i,j} \to \mathbb{R}_{\geq 0}$ *to be the message from the function node* $f_{\mathrm{r},j}$ *on the RHS to the function node* $f_{\mathrm{l},i}$ *on*

the LHS, and define a right-going message $\vec{\mu}_{t,i,j} : \mathcal{X}_{i,j} \to \mathbb{R}_{\geq 0}$ to be the message from the function node $f_{\mathrm{l},i}$ on the LHS to the function node $f_{\mathrm{r},j}$ on the RHS.

- *It is sufficient to keep track of the inverse likelihood ratios in the SPA:*

$$\overleftarrow{V}_t(i,j) \triangleq \frac{\overleftarrow{\mu}_{t,i,j}(1)}{\overleftarrow{\mu}_{t,i,j}(0)}, \quad \vec{V}_t(i,j) \triangleq \frac{\vec{\mu}_{t,i,j}(1)}{\vec{\mu}_{t,i,j}(0)}, \quad (i,j) \in \mathcal{E}.$$

- *Define* $\overleftarrow{\boldsymbol{V}}_t \triangleq \big(\overleftarrow{V}_t(i,j)\big)_{(i,j)}$ *and* $\vec{\boldsymbol{V}}_t \triangleq \big(\vec{V}_t(i,j)\big)_{(i,j)}$.

Lemma 18. *[11, Lemma 29] The SPA update rules on* N *with respect to the inverse likelihood ratios are given as follows.*

1) *For* $t = 0$, *initialize* $0 < \overleftarrow{V}_0(i,j) < \infty$ *for all* $(i,j) \in \mathcal{E}$.
2) *For* $t \in \mathbb{Z}_{\geq 1}$ *and* $(i,j) \in \mathcal{E}$, *the update rules for the inverse likelihood ratios are given by*

$$\vec{V}_t(i,j) = \frac{\sum\limits_{\boldsymbol{x}(i,:):\, x(i,j)=1} f_{\mathrm{l},i}\big(\boldsymbol{x}(i,:)\big) \cdot \prod\limits_{j_1 \in \mathcal{J}_i \setminus \{j\}} \big(\overleftarrow{V}_{t-1}(i,j_1)\big)^{x(i,j_1)}}{\sum\limits_{\boldsymbol{x}(i,:):\, x(i,j)=0} f_{\mathrm{l},i}\big(\boldsymbol{x}(i,:)\big) \cdot \big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)^{\boldsymbol{x}(i,:)}},$$

$$\overleftarrow{V}_t(i,j) = \frac{\sum\limits_{\boldsymbol{x}(:,j):\, x(i,j)=1} f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \cdot \prod\limits_{i_1 \in \mathcal{I}_j \setminus \{i\}} \big(\vec{V}_t(i_1,j)\big)^{x(i_1,j)}}{\sum\limits_{\boldsymbol{x}(:,j):\, x(i,j)=0} f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \cdot \big(\vec{\boldsymbol{V}}_t(:,j)\big)^{\boldsymbol{x}(:,j)}}.$$

3) *The SPA update rules above can be written as* $(\overleftarrow{\boldsymbol{V}}_t, \vec{\boldsymbol{V}}_t) = f_{\mathrm{SPA,m}}(\overleftarrow{\boldsymbol{V}}_{t-1})$ *for some suitably defined function* $f_{\mathrm{SPA,m}}$, *where the index "m" means that it is related to the messages update rules.*

The collection of beliefs $\boldsymbol{\beta}_t = \big((\boldsymbol{\beta}_{i,t})_i, (\boldsymbol{\beta}_{j,t})_j, (\beta_{\mathcal{E},t}(i,j))_{i,j}\big)$ *evaluated at the* t*-th iteration is given by*

$$\beta_{i,t}\big(\boldsymbol{x}(i,:)\big) = \frac{f_{\mathrm{l},i}\big(\boldsymbol{x}(i,:)\big) \cdot \big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)^{\boldsymbol{x}(i,:)}}{p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)}, \ \boldsymbol{x}(i,:) \in \mathcal{X}_{f_{\mathrm{l},i}},$$

$$\beta_{j,t}\big(\boldsymbol{x}(:,j)\big) = \frac{f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \cdot \big(\vec{\boldsymbol{V}}_t(:,j)\big)^{\boldsymbol{x}(:,j)}}{q_j\big(\vec{\boldsymbol{V}}_t(:,j)\big)}, \ \boldsymbol{x}(:,j) \in \mathcal{X}_{f_{\mathrm{r},j}},$$

$$\beta_{\mathcal{E},t}(i,j) = \frac{\overleftarrow{V}_{t-1}(i,j) \cdot \vec{V}_t(i,j)}{1 + \overleftarrow{V}_{t-1}(i,j) \cdot \vec{V}_t(i,j)},$$

where p_i *and* q_j, *as defined in Definition 3, ensure that* $\sum_{\boldsymbol{x}(i,:)} \beta_{i,t}\big(\boldsymbol{x}(i,:)\big) = \sum_{\boldsymbol{x}(:,j)} \beta_{j,t}\big(\boldsymbol{x}(:,j)\big) = 1$. *Similarly, the above expressions for* $\boldsymbol{\beta}_t$ *can be written as* $\boldsymbol{\beta}_t = f_{\mathrm{SPA,b}}(\overleftarrow{\boldsymbol{V}}_{t-1})$ *for some suitably defined function* $f_{\mathrm{SPA,b}}$, *where the index "b" means that it is related to the evaluation of the beliefs.*

Definition 19. *We define* $\overleftarrow{\boldsymbol{V}} \triangleq \big(\overleftarrow{V}(i,j)\big)_{(i,j)}$ *and* $\vec{\boldsymbol{V}} \triangleq \big(\vec{V}(i,j)\big)_{(i,j)}$. *Then we say a collection of inverse likelihood ratios* $(\overleftarrow{\boldsymbol{V}}, \vec{\boldsymbol{V}})$ *constitutes an SPA fixed point if all the ratios are positive-valued*[3] *and* $(\overleftarrow{\boldsymbol{V}}, \vec{\boldsymbol{V}}) = f_{\mathrm{SPA,m}}(\overleftarrow{\boldsymbol{V}})$. *The collection of beliefs* $\boldsymbol{\beta}$ *evaluated at this SPA fixed point is given by* $\boldsymbol{\beta} = f_{\mathrm{SPA,b}}(\overleftarrow{\boldsymbol{V}})$.

We can use the so-called pseudo-dual function of the Bethe free energy function (see, *e.g.*, [31, Theorem 4] and [1, Eq. (14.27)]) to track the behavior of the SPA.

[3]In general, the inverse likelihood ratio messages at an SPA fixed point can take value in $\{\mathbb{R}_{\geq 0} \cup \{\infty\}\}^{|\mathcal{E}|}$. In this paper, due to the space constraints, we focus on the typical case where the inverse likelihood ratios that constitute an SPA fixed point are positive-valued.

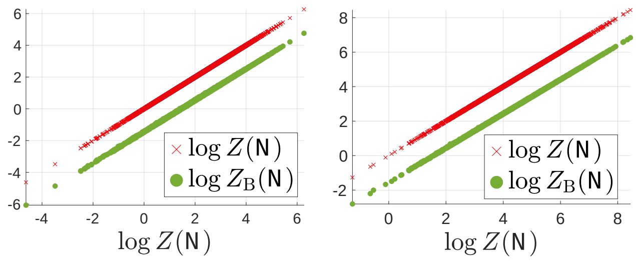

Fig. 2: The numerical results in Example 24.

Lemma 20. *For any collection of inverse likelihood ratios* $(\overleftarrow{\boldsymbol{V}}, \vec{\boldsymbol{V}})$, *the pseudo-dual function of the Bethe free energy function is given by*

$$F_{\mathrm{B},\#}(\overleftarrow{\boldsymbol{V}}, \vec{\boldsymbol{V}}) = -\sum_i \log\Big(p_i\big(\overleftarrow{\boldsymbol{V}}(i,:)\big)\Big) - \sum_j \log\Big(q_j\big(\vec{\boldsymbol{V}}(:,j)\big)\Big) + \sum_{(i,j)} \log\Big(1 + \overleftarrow{V}(i,j) \cdot \vec{V}(i,j)\Big).$$

Proof. The proof is a straightforward generalization of the proof of [11, Lemma 31] and thus it is omitted here. ■

Proposition 21. *If a stationary point of* $F_{\mathrm{B},\#}(\overleftarrow{\boldsymbol{V}}, \vec{\boldsymbol{V}})$ *satisfies* $\overleftarrow{\boldsymbol{V}}, \vec{\boldsymbol{V}} \in \mathbb{R}_{>0}^{|\mathcal{E}|}$, *then it corresponds to an SPA fixed point of* N.

Proof. This follows from the definition of $F_{\mathrm{B},\#}$ and the definition of the SPA fixed point. The details are omitted here. ■

The following theorem states that the Bethe partition function can be evaluated by the collection of inverse likelihood ratios $(\overleftarrow{\boldsymbol{V}}, \vec{\boldsymbol{V}})$ that constitutes an SPA fixed point.

Theorem 22. *For each collection of positive-valued inverse likelihood ratios* $(\overleftarrow{\boldsymbol{V}}, \vec{\boldsymbol{V}})$ *that constitutes an SPA fixed point, the beliefs given by* $\boldsymbol{\beta} = f_{\mathrm{SPA,b}}(\overleftarrow{\boldsymbol{V}})$ *are the location of the minimum of the Bethe free energy function, and the Bethe partition function is given by* $Z_{\mathrm{B}}(\mathsf{N}) = \exp\big(-F_{\mathrm{B},\#}(\overleftarrow{\boldsymbol{V}}, \vec{\boldsymbol{V}})\big)$.

Proof. See Appendix G. ■

The following theorem gives a non-trivial generalization of the result in [11, Theorem 32].

Theorem 23. *Consider the S-NFG* N *such that there is an SPA fixed point consisting of positive-valued inverse likelihood ratio messages only. The SPA converges exponentially fast to this SPA fixed point.*

Proof. See Appendix H. ■

In the following example, we provide numerical results comparing $Z(\mathsf{N})$ with $Z_{\mathrm{B}}(\mathsf{N})$ for some small S-NFGs.

Example 24. *We consider the example S-NFG as defined in Example 5, which is defined based on a matrix* $\boldsymbol{\theta} \in \mathbb{R}_{>0}^{n \times m}$.

We first consider the case $n = m = 6$ *and* $r_i = c_j = 2$ *for all* $i \in \mathcal{I}$ *and* $j \in \mathcal{J}$. *We randomly generate* 3000 *instances of* $\boldsymbol{\theta}$, *where in each instance the entries of* $\boldsymbol{\theta}$ *are randomly generated i.i.d. according to the uniform distribution in the interval* $(0,1)$. *Fig. 2(left) shows the obtained* $Z(\mathsf{N})$ *and* $Z_{\mathrm{B}}(\mathsf{N})$. *We can see that* $Z_{\mathrm{B}}(\mathsf{N})$ *lower bounds* $Z(\mathsf{N})$, *corroborating Theorem 11, and that* $Z_{\mathrm{B}}(\mathsf{N})$ *provides a good estimate of* $Z(\mathsf{N})$ *in this case.*

Consider the same setup as the previous case, but with $n = m = 6$ *replaced by* $n = m = 7$. *The obtained numerical results are presented in Fig. 2(right). We can make similar observations about the values of* $Z(\mathsf{N})$ *and* $Z_{\mathrm{B}}(\mathsf{N})$ *as in the previous case.*

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Appendix A
Real Stable Polynomials

In this appendix, we introduce real stable polynomials and discuss some of their properties. Throughout this appendix, we will assume that $n \in \mathbb{Z}_{\geq 1}$, *i.e.*, n is a positive integer greater than zero.

Definition 25. *Let $\mathbb{C}[z(1), \ldots, z(n)]$ denote the set of polynomials that consist of complex-valued coefficients and variables $\boldsymbol{z} \triangleq \big(z(1), \ldots, z(n)\big) \in \mathbb{C}^n$. We make the following definitions for a polynomial $h \in \mathbb{C}[z(1), \ldots, z(n)]$.*

1) *Suppose that there is a finite set $\mathcal{A}_h \subseteq \mathbb{Z}^n_{\geq 0}$ with elements denoted by $\boldsymbol{\alpha} \triangleq \big(\alpha(1), \ldots, \alpha(n)\big) \in \mathcal{A}_h$ and a mapping $a_h : \mathcal{A}_h \to \mathbb{C} \setminus \{0\}$, $\boldsymbol{\alpha} \mapsto a_h(\boldsymbol{\alpha})$ such that*
$$h(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}_h} a_h(\boldsymbol{\alpha}) \cdot \boldsymbol{z}^{\boldsymbol{\alpha}}.$$
2) *We say that $h \in \mathbb{R}[z(1), \ldots, z(n)]$ if $a_h(\boldsymbol{\alpha}) \in \mathbb{R}$ for all $\boldsymbol{\alpha} \in \mathcal{A}_h$.*
3) *We say that $h \in \mathbb{R}_{\geq 0}[z(1), \ldots, z(n)]$ if $a_h(\boldsymbol{\alpha}) \in \mathbb{R}_{\geq 0}$ for all $\boldsymbol{\alpha} \in \mathcal{A}_h$.*
4) *The degree of the polynomial h, denoted by d_h, is defined to be*
$$d_h \triangleq \max_{\boldsymbol{\alpha} \in \mathcal{A}_h} \sum_{i \in [n]} \alpha(i) \in \mathbb{Z}_{\geq 0}.$$
5) *We say that the polynomial h is homogeneous if $\sum_{i \in [n]} \alpha(i) = d_h$ for all $\boldsymbol{\alpha} \in \mathcal{A}_h$.*
6) *We say that the polynomial h is a multi-affine polynomial if $\boldsymbol{\alpha} \in \{0, 1\}^n$ for all $\boldsymbol{\alpha} \in \mathcal{A}_h$.*
7) *We define*
$$\mathrm{supp}(h) \triangleq \{\boldsymbol{\alpha} \in \mathcal{A}_h \mid a_h(\boldsymbol{\alpha}) \neq 0\}.$$
to be the support of h. From the above assumptions, it is straightforward to see that $\mathrm{supp}(h) = \mathcal{A}_h$.

Definition 26. *We say that a polynomial $h \in \mathbb{C}[z(1), \ldots, z(n)]$ is $\mathcal{H}$-stable if $h(\boldsymbol{z}) \neq 0$ for all $\boldsymbol{z} \in \mathcal{H}^n$, where*
$$\mathcal{H} \triangleq \{c \in \mathbb{C} \mid \mathrm{Im}(c) > 0\}$$
and $\mathrm{Im}(c)$ gives the imaginary part of $c \in \mathbb{C}$. We further say that $h(\boldsymbol{z})$ is real stable if $h \in \mathbb{R}[z(1), \ldots, z(n)]$ and h is $\mathcal{H}$-stable.

Lemma 27. *Let $h(\boldsymbol{z}) \in \mathbb{R}[z(1), \ldots, z(n)]$ be a real stable polynomial. Then for any $\boldsymbol{b} \in \mathbb{R}^n_{\geq 0}$, the polynomial $h(\boldsymbol{b} \cdot \boldsymbol{z})$ is also a real stable polynomial with respect to $\boldsymbol{z} \in \mathbb{C}^n$.*

Proof. This result follows straightforwardly from the definition of real stable polynomials. ■

Lemma 28. *Let $h(\boldsymbol{z}) \in \mathbb{R}[z(1), \ldots, z(n)]$ be a real stable polynomial. Then for any $\boldsymbol{b} \in \mathbb{R}^n$, the polynomial $h(\boldsymbol{b} - \boldsymbol{z})$ is also a real stable polynomial with respect to $\boldsymbol{z} \in \mathbb{C}^n$.*

Proof. We prove this claim by contradiction. Fix a vector $\boldsymbol{b} \in \mathbb{R}^n$. Suppose that there exists a vector $\boldsymbol{z}'$ such that
$$h(\boldsymbol{b} - \boldsymbol{z}') = 0, \quad \mathrm{Im}(z'_1), \ldots, \mathrm{Im}(z'_n) > 0.$$
Then we have
$$0 = \overline{h(\boldsymbol{b} - \boldsymbol{z}')} = h(\boldsymbol{b} - \overline{\boldsymbol{z}'}),$$
where the second equality follows from $h \in \mathbb{R}[z(1), \ldots, z(n)]$ and $\boldsymbol{b} \in \mathbb{R}^n$. Based on that, we have
$$h(\boldsymbol{b} - \overline{\boldsymbol{z}'}) = 0, \quad \mathrm{Im}(-\overline{z'_1}), \ldots, \mathrm{Im}(-\overline{z'_n}) > 0,$$
which is a contradiction to the fact that h is real stable. ■

Lemma 29. *[32] Let $h \in \mathbb{R}_{\geq 0}[z(1), \ldots, z(n)]$ be a real stable polynomial. Then $\log\big(h(\boldsymbol{z})\big)$ is a concave function with respect to $\boldsymbol{z} \in \mathbb{R}^n_{\geq 0}$.*

For a more accessible proof of Lemma 29, see [33, Remark 3.8].

Theorem 30. *[22, Theorem 5.6] A multi-affine polynomial $h(\boldsymbol{z}) \in \mathbb{R}[z(1), \ldots, z(n)]$ is real stable if and only if for all $\boldsymbol{z}' \in \mathbb{R}^n$ and $i, j \in [n]$ we have*
$$\frac{\partial}{\partial z_i} h(\boldsymbol{z})\bigg|_{\boldsymbol{z}=\boldsymbol{z}'} \cdot \frac{\partial}{\partial z_j} h(\boldsymbol{z})\bigg|_{\boldsymbol{z}=\boldsymbol{z}'} - h(\boldsymbol{z}) \cdot \frac{\partial^2}{\partial z_i \, \partial z_j} h(\boldsymbol{z})\bigg|_{\boldsymbol{z}=\boldsymbol{z}'} \geq 0.$$

Appendix B
Proof of Theorem 8

In this appendix, we prove Theorem 8, *i.e.*, the equality $\mathrm{conv}(\mathcal{C}) = \mathcal{L}_{\mathcal{E}}$, using results from matroid theory [34]. We begin by introducing key concepts from matroid theory that are relevant to our proof.

Definition 31. *[34, Section 1.1] A finite matroid M is a pair $(\mathcal{G}, \mathcal{B})$, where $\mathcal{G}$ is a finite set, called the ground set, and $\mathcal{B}$ is a collection of subsets of $\mathcal{G}$, called bases, with the following properties.*

1) *$\mathcal{B}$ is non-empty.*
2) *If both $\mathcal{B}_1$ and $\mathcal{B}_2$ are distinct members of $\mathcal{B}$ and $x_1 \in \mathcal{B}_1 \setminus \mathcal{B}_2$, then there exists an $x_2 \in \mathcal{B}_2 \setminus \mathcal{B}_1$ such that $\big(\mathcal{B}_1 \setminus \{x_1\}\big) \cup \{x_2\} \in \mathcal{B}$.*

Based on the definition of M, we make further definitions.

3) *Because $|\mathcal{B}_1| = |\mathcal{B}_2|$ for all $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}$, we define the rank $r(M)$ of the matroid M to be $r(M) \triangleq |\mathcal{B}_1|$ for arbitrary $\mathcal{B}_1 \in \mathcal{B}$.*
4) *A set $\mathcal{K}_1 \subseteq \mathcal{G}$ is called an independent set if it is a subset of one of the bases, i.e., there exists $\mathcal{B}_1 \in \mathcal{B}$ such that $\mathcal{K}_1 \subseteq \mathcal{B}_1$.*
5) *The collection of subsets of $\mathcal{G}$ that consists of all the independent sets, is called the family of independent sets and is denoted by $\mathcal{K}$. One can also define M based on the pair $(\mathcal{G}, \mathcal{K})$.*

Definition 32. *Consider the S-NFG N and the associated set of edges $\mathcal{E}$ as defined in Definition 3. Let M be a matroid with the ground set given by $\mathcal{E}$. For any base $\mathcal{E}' \subseteq \mathcal{E}$ of M, define*
$$\boldsymbol{x}_{\mathcal{E}'} \triangleq \big(x_{\mathcal{E}'}(i,j)\big)_{(i,j)} \in \mathcal{X}$$
with
$$x_{\mathcal{E}'}(i,j) \triangleq \begin{cases} 1 & \text{if } (i,j) \in \mathcal{E}' \\ 0 & \text{otherwise} \end{cases}.$$
The associated matroid polytope is defined to be
$$\mathcal{P}(M) \triangleq \mathrm{conv}\big(\{\boldsymbol{x}_{\mathcal{E}'} \mid \mathcal{E}' \text{ is a base of } M\}\big).$$

Lemma 33. *[22, Corollary 3.4] For any multi-affine and homogeneous $\mathcal{H}$-stable polynomial $h \in \mathbb{C}[z(1), \ldots, z(n)]$, its support $\mathrm{supp}(h)$ is the set of indicator vectors for the set of bases of a matroid.*

Remark 34. *As proven in [22, Theorem 6.6], there exist matroids, e.g., the Fano matroid, such that no $\mathcal{H}$-stable polynomial has support corresponding to the set of bases of this matroid.*

In this paper, when we state that the support of a polynomial corresponds to the set of bases of a matroid, we mean that this support is the set of indicator vectors corresponding to the set of bases of this matroid.

Definition 35. *Consider the bipartite S-NFG* N *defined in Definition 3. By Lemma 33 and the definition of the local functions in Definition 3, we know that the support of each local function corresponds to the set of bases of a matroid, which motivates the following definitions.*

1) *Define* M_{l} *to be a matroid with the ground set* $\mathcal{E}$ *and the set of bases:*

$$\mathcal{B}_{\mathrm{l}} \triangleq \{\mathcal{E}' \subseteq \mathcal{E} \mid \boldsymbol{x}_{\mathcal{E}'}(i,:) \in \mathcal{X}_{f_{\mathrm{l},i}}, \forall i \in \mathcal{I}\},$$

where we have used $\boldsymbol{x}_{\mathcal{E}'}$ *in Definition 32.*

2) *Similarly, define* M_{r} *to be a matroid with the ground set* $\mathcal{E}$ *and the set of bases:*

$$\mathcal{B}_{\mathrm{r}} \triangleq \{\mathcal{E}' \subseteq \mathcal{E} \mid \boldsymbol{x}_{\mathcal{E}'}(:,j) \in \mathcal{X}_{f_{\mathrm{r},j}}, \forall j \in \mathcal{J}\}.$$

Note that $\mathcal{B}_{\mathrm{l}}$ can be viewed as a direct product of the sets of bases of the matroids associated with the local functions on the LHS of N, *i.e.*, the local functions in $\{f_{\mathrm{l},i}\}_{i\in\mathcal{I}}$. Similarly, $\mathcal{B}_{\mathrm{r}}$ is a direct product of the sets of bases of the matroids associated with the local functions on the RHS of N, *i.e.*, the local functions in $\{f_{\mathrm{r},j}\}_{j\in\mathcal{J}}$.

Lemma 36. *The matroid polytopes of* M_{l} *and* M_{r}*, denoted by* $\mathcal{P}(M_{\mathrm{l}})$ *and* $\mathcal{P}(M_{\mathrm{r}})$*, are given by*

$$\mathcal{P}(M_{\mathrm{l}}) = \{\boldsymbol{\beta}_{\mathcal{E}} \mid \boldsymbol{\beta}_{\mathcal{E}}(i,:) \in \mathrm{conv}(\mathcal{X}_{f_{\mathrm{l},i}}), \forall i \in \mathcal{I}\}, \quad (2)$$
$$\mathcal{P}(M_{\mathrm{r}}) = \{\boldsymbol{\beta}_{\mathcal{E}} \mid \boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathrm{conv}(\mathcal{X}_{f_{\mathrm{r},j}}), \forall j \in \mathcal{J}\}. \quad (3)$$

Proof. This follows from the definition of the matroid polytope in Definition 32 and the definition of the matroids M_{r} and M_{l} in Definition 35. ∎

Definition 37. *Consider* M_{l} *and* M_{r} *defined in Definition 35. Define*

$$\mathcal{P}(M_{\mathrm{r}} \cap M_{\mathrm{l}}) \triangleq \mathrm{conv}(\{\boldsymbol{x}_{\mathcal{E}'} \mid \mathcal{E}' \in \mathcal{B}_{\mathrm{l}} \cap \mathcal{B}_{\mathrm{r}}\}).$$

Consider two matroids with the same ground set but different families of independent sets: $M_1 = (\mathcal{G}, \mathcal{K}_1)$ and $M_2 = (\mathcal{G}, \mathcal{K}_2)$ with $\mathcal{K}_1 \neq \mathcal{K}_2$. Although the pair $(\mathcal{G}, \mathcal{K}_1 \cap \mathcal{K}_2)$ is not a matroid in general, one can show the following result.

Theorem 38. *Consider* M_{l} *and* M_{r} *defined in Definition 35. All the vertices in the polytope* $\mathcal{P}(M_{\mathrm{r}}) \cap \mathcal{P}(M_{\mathrm{l}})$ *are in the set* $\mathcal{B}_{\mathrm{l}} \cap \mathcal{B}_{\mathrm{r}}$*, i.e.,*

$$\mathcal{P}(M_{\mathrm{l}}) \cap \mathcal{P}(M_{\mathrm{r}}) = \mathcal{P}(M_{\mathrm{l}} \cap M_{\mathrm{r}}).$$

Proof. This is a corollary of the results in [23]. For a more accessible proof, see [35, Corollary 41.12b]. ∎

Now we prove Theorem 8 using the previously introduced concepts and results from matroid theory. It holds that

$$\begin{aligned}
\mathrm{conv}(\mathcal{C}) &\overset{(a)}{=} \mathrm{conv}\left(\left\{\boldsymbol{x} \in \mathcal{X} \;\middle|\; \begin{array}{l} \boldsymbol{x}(i,:) \in \mathcal{X}_{f_{\mathrm{l},i}}, \forall i \in \mathcal{I} \\ \boldsymbol{x}(:,j) \in \mathcal{X}_{f_{\mathrm{r},j}}, \forall j \in \mathcal{J} \end{array}\right\}\right) \\
&\overset{(b)}{=} \mathrm{conv}(\{\boldsymbol{x}_{\mathcal{E}'} \mid \mathcal{E}' \in \mathcal{B}_{\mathrm{l}} \cap \mathcal{B}_{\mathrm{r}}\}) \\
&\overset{(c)}{=} \mathcal{P}(M_{\mathrm{l}} \cap M_{\mathrm{r}}) \\
&\overset{(d)}{=} \mathcal{P}(M_{\mathrm{l}}) \cap \mathcal{P}(M_{\mathrm{r}}) \\
&\overset{(e)}{=} \left\{\boldsymbol{\beta}_{\mathcal{E}} \;\middle|\; \begin{array}{l} \boldsymbol{\beta}_{\mathcal{E}}(i,:) \in \mathrm{conv}(\mathcal{X}_{f_{\mathrm{l},i}}), \forall i \in \mathcal{I} \\ \boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathrm{conv}(\mathcal{X}_{f_{\mathrm{r},j}}), \forall j \in \mathcal{J} \end{array}\right\} \\
&\overset{(f)}{=} \mathcal{L}_{\mathcal{E}},
\end{aligned}$$

- where step (a) follows from the definition of the set of the valid configurations of N in Definition 3,
- where step (b) follows from the definition of the sets of bases $\mathcal{B}_{\mathrm{l}}$ and $\mathcal{B}_{\mathrm{r}}$ in Definition 35,
- where step (c) follows from the definition of the polytope in Definition 37,
- where step (d) follows from Theorem 38,
- where step (e) follows from the expressions for the sets $\mathcal{P}(M_{\mathrm{l}})$ and $\mathcal{P}(M_{\mathrm{r}})$ in Lemma 36,
- where step (f) follows from the definition of $\mathcal{L}_{\mathcal{E}}$ in Definition 6.

Appendix C
Relating the Supports of the Local Functions in Example 5 to Uniform Matroids

Motivated by Lemma 33, in this appendix, we show that the S-NFG in Example 5 corresponds to uniform matroids. For the definition of a matroid, see Definition 31.

Definition 39. ***(Uniform Matroid)*** *[34, Example 1.2.7] Consider an arbitrary integer* $r \in \mathbb{Z}_{\geq 1}$ *and an arbitrary finite ground set* $\mathcal{G}$ *such that* $r \leq |\mathcal{G}|$*. A uniform matroid of rank* r *is defined to be a matroid* $M = (\mathcal{G}, \mathcal{B})$ *such that the set of bases is*

$$\mathcal{B} = \{\mathcal{B}_1 \subseteq \mathcal{G} \mid |\mathcal{B}_1| = r\}.$$

Notice that for the S-NFG in Example 5, the supports of the local functions are given by

$$\mathcal{X}_{f_{\mathrm{l},i}} = \{\boldsymbol{x}(i,:) \in \{0,1\}^{m_i} \mid w_{\mathrm{H}}(\boldsymbol{x}(i,:)) = r_i\}, \quad i \in \mathcal{I},$$
$$\mathcal{X}_{f_{\mathrm{r},j}} = \{\boldsymbol{x}(:,j) \in \{0,1\}^{n_j} \mid w_{\mathrm{H}}(\boldsymbol{x}(:,j)) = c_j\}, \quad j \in \mathcal{J}.$$

For each $i \in \mathcal{I}$, consider a matroid $M_{f_{\mathrm{l},i}}$ with the ground set $\mathcal{J}_i$ and the set of bases

$$\{\mathcal{J}_i' \subseteq \mathcal{J}_i \mid |\mathcal{J}_i'| = r_i\}.$$

Then $M_{f_{\mathrm{l},i}}$ is a uniform matroid of rank r_i, and the set of the indicator vectors that corresponds to the set of bases of $M_{f_{\mathrm{l},i}}$, is given by $\mathcal{X}_{f_{\mathrm{l},i}}$.

Similarly, for each $j \in \mathcal{J}$, the set $\mathcal{X}_{f_{\mathrm{r},j}}$ corresponds to the set of bases of a uniform matroid of rank c_j.

Appendix D
Proof of Theorem 14

For any $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$ and $i \in \mathcal{I}$, we get

$$\begin{aligned}
&\inf_{\boldsymbol{L}(i,:)\in\mathbb{R}^{m_i}} \frac{\tilde{p}_i(\boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:))}{\exp(\langle \boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:)\rangle)} \\
&\overset{(a)}{=} \inf_{\boldsymbol{L}'(i,:)\in\mathbb{R}^{m_i}_{>0}} \frac{\sum\limits_{\boldsymbol{x}(i,:)\in\{0,1\}^{m_i}} f_{\mathrm{l},i}(\boldsymbol{x}(i,:)) \cdot (\boldsymbol{L}'(i,:) \cdot \boldsymbol{\beta}_{\mathcal{E}}(i,:))^{\boldsymbol{x}(i,:)}}{(\boldsymbol{L}'(i,:))^{\boldsymbol{\beta}_{\mathcal{E}}(i,:)}} \\
&\overset{(b)}{=} (\boldsymbol{\beta}_{\mathcal{E}}(i,:))^{\boldsymbol{\beta}_{\mathcal{E}}(i,:)} \cdot \inf_{\boldsymbol{L}''(i,:)\in\mathbb{R}^{m_i}_{>0}} \frac{p_i(\boldsymbol{L}''(i,:))}{(\boldsymbol{L}''(i,:))^{\boldsymbol{\beta}_{\mathcal{E}}(i,:)}}, \qquad (4)
\end{aligned}$$

where step (a) follows from the substitution

$$\boldsymbol{L}'(i,:) = \exp(\boldsymbol{L}(i,:)) \in \mathbb{R}^{m_i}_{>0}, \qquad \boldsymbol{L}(i,:) \in \mathbb{R}^{m_i},$$

and where step (b) follows from [15, Lemma 2.12]. Similarly, for any $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$ and $j \in \mathcal{J}$, we have

$$\begin{aligned}
&\inf_{\boldsymbol{R}(:,j)\in\mathbb{R}^{n_j}} \frac{\tilde{q}_j(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j))}{\exp(\langle \boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\rangle)} \\
&\overset{(a)}{=} \inf_{\boldsymbol{R}'(:,j)\in\mathbb{R}^{n_j}_{>0}} \left\{ \frac{1}{(\boldsymbol{R}'(:,j))^{\boldsymbol{\beta}_{\mathcal{E}}(:,j)}} \right. \\
&\qquad \left. \cdot \sum_{\boldsymbol{x}(:,j)\in\{0,1\}^{m_i}} f_{\mathrm{r},j}(\boldsymbol{x}(:,j)) \cdot (\boldsymbol{R}'(:,j))^{\boldsymbol{x}(:,j)} \cdot (\boldsymbol{1} - \boldsymbol{\beta}_{\mathcal{E}}(:,j))^{\boldsymbol{1}-\boldsymbol{x}(:,j)} \right\}
\end{aligned}$$

$$
\begin{aligned}
Z_{\mathrm{B}}(\mathsf{N}) &\overset{(a)}{=} \max_{\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}} \big(\boldsymbol{\beta}_{\mathcal{E}}\big)^{\boldsymbol{\beta}_{\mathcal{E}}} \cdot \big(\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}\big)^{\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}} \cdot \inf_{\boldsymbol{L}'', \boldsymbol{R}'' \in \mathbb{R}_{>0}^{|\mathcal{E}|}} \frac{\left(\prod_i p_i(\boldsymbol{L}''(i,:))\right) \cdot \left(\prod_j q_j(\boldsymbol{R}''(:,j))\right)}{\big(\boldsymbol{L}'' \cdot \boldsymbol{R}''\big)^{\boldsymbol{\beta}_{\mathcal{E}}}} \\
&\overset{(b)}{=} \max_{\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}} \inf_{\boldsymbol{L}, \boldsymbol{R} \in \mathbb{R}^{|\mathcal{E}|}} \frac{\left(\prod_i \tilde{p}_i\big(\boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:)\big)\right) \cdot \left(\prod_j \tilde{q}_j\big(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big)\right)}{\exp(\langle \boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}+\boldsymbol{R}\rangle)} \\
&\overset{(c)}{=} \tilde{Z}_{\mathrm{B,d}}(\mathsf{N}).
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\overset{(b)}{=} \big(\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j)\big)^{-\boldsymbol{\beta}_{\mathcal{E}}(:,j)} \cdot \inf_{\boldsymbol{R}'(:,j) \in \mathbb{R}_{>0}^{n_j}} \Bigg\{ \frac{1}{\big(\boldsymbol{R}'(:,j)\big)^{\boldsymbol{\beta}_{\mathcal{E}}(:,j)}} \\
&\quad \cdot \sum_{\boldsymbol{x}(:,j) \in \{0,1\}^{m_i}} f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \cdot \big(\boldsymbol{R}'(:,j)\big)^{\boldsymbol{x}(:,j)} \cdot \big(\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j)\big)^{\mathbf{1}} \Bigg\} \\
&= \big(\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j)\big)^{\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j)} \cdot \inf_{\boldsymbol{R}''(:,j) \in \mathbb{R}_{>0}^{n_j}} \frac{q_j\big(\boldsymbol{R}''(:,j)\big)}{\big(\boldsymbol{R}''(:,j)\big)^{\boldsymbol{\beta}_{\mathcal{E}}(:,j)}},
\end{aligned}
\tag{5}
$$

where step (a) follows from the substitution:

$$
\boldsymbol{R}'(:,j) = \exp(\boldsymbol{R}(:,j)) \in \mathbb{R}_{>0}^{n_j}, \qquad \boldsymbol{R}(:,j) \in \mathbb{R}^{n_j},
$$

and where step (b) follows from [15, Lemma 2.12]. We then obtain the equalities in (7) at the top of this page, where step (a) follows from [20, Theorem 3.1], where step (b) follows from the equalities in (4) and (5), and where step (c) follows from the definition of $\tilde{Z}_{\mathrm{B,d}}$ in Definition 13.

Appendix E
Proof of Theorem 15

We first present some properties for the polynomials $\tilde{p}_i$ and $\tilde{q}_j$ for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$.

Lemma 40. *For any $\boldsymbol{L}, \boldsymbol{R} \in \mathbb{R}^{|\mathcal{E}|}$, the following two properties hold.*

1) *For each $i \in \mathcal{I}$, the polynomial $\tilde{p}_i\big(\boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:)\big)$ is a real stable polynomial with respect to $\boldsymbol{\beta}_{\mathcal{E}}(i,:) \in \mathbb{C}^{m_i}$.*
2) *For each $j \in \mathcal{J}$, the polynomial $\tilde{q}_j\big(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big)$ is a real stable polynomial with respect to $\boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathbb{C}^{n_j}$.*

Proof. We prove the second statement first. Fix $j \in \mathcal{J}$. By the definition of N in Definition 3, we know that

$$
q_j(\boldsymbol{\beta}_{\mathcal{E}}(:,j)) \neq 0, \qquad \boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathcal{H}^{n_j}.
$$

Because the imaginary parts of $\beta_{\mathcal{E}}(i,j)$ and $(1-\beta_{\mathcal{E}}(i,j))^{-1}$ have the same sign, we have

$$
\begin{aligned}
q_j\Big(\big(\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j)\big)^{-1}\Big) &= \sum_{\boldsymbol{x}(:,j)} f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \cdot \big(\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j)\big)^{-\boldsymbol{x}(:,j)} \\
&\neq 0, \qquad \boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathcal{H}^{n_j}.
\end{aligned}
$$

Because

$$
\begin{aligned}
\exp\big(R(i,j)\big) &\in \mathbb{R}_{>0}, \qquad i \in \mathcal{I}_j,\ \boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathcal{H}^{n_j}, \\
1-\beta_{\mathcal{E}}(i,j) &\neq 0, \qquad i \in \mathcal{I}_j,\ \boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathcal{H}^{n_j},
\end{aligned}
$$

by Lemma 27, we have

$$
\begin{aligned}
&\tilde{q}_j\big(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big) \\
&= \left(\prod_{i \in \mathcal{I}_j} \big(1-\beta_{\mathcal{E}}(i,j)\big)\right) \cdot q_j\Big(\exp\big(\boldsymbol{R}(:,j)\big) \cdot \big(\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j)\big)^{-1}\Big) \\
&\neq 0, \qquad \boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathcal{H}^{n_j},
\end{aligned}
$$

which proves the real stability of $\tilde{q}_j$ with respect to the vector $\boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathbb{C}^{m_i}$.

For each $i \in \mathcal{I}$, proving that $\tilde{p}_i\big(\boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:)\big)$ is real stable with respect to $\boldsymbol{\beta}_{\mathcal{E}}(i,:) \in \mathbb{C}^{m_i}$ for fixed $\boldsymbol{L}(i,:) \in \mathbb{R}^{|\mathcal{J}_i|}$ is actually easier and thus it is omitted here. ∎

Lemma 41. *For any $\boldsymbol{L}, \boldsymbol{R} \in \mathbb{R}^{|\mathcal{E}|}$, the following two properties hold.*

1) *For each $i \in \mathcal{I}$, the function*
$$
\log\Big(\tilde{p}_i\big(\boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:)\big)\Big)
$$
is a concave function with respect to $\boldsymbol{\beta}_{\mathcal{E}}(i,:) \in [0,1]^{m_i}$.
2) *For each $j \in \mathcal{J}$, the function*
$$
\log\Big(\tilde{q}_j\big(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big)\Big)
$$
is a concave function with respect to $\boldsymbol{\beta}_{\mathcal{E}}(:,j) \in [0,1]^{n_j}$.

Proof. We prove the second statement first. We consider that both $j \in \mathcal{J}$ and $\boldsymbol{R}(:,j) \in \mathbb{R}^{m_i}$ are fixed. Note that the polynomial $\tilde{q}_j\big(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big)$ is not in the set $\mathbb{R}_{\geq 0}[\boldsymbol{\beta}_{\mathcal{E}}(:,j)]$ and we cannot simply use Lemma 29 to prove the log-concavity of this polynomial. By Lemmas 28 and 40, the polynomial

$$
\begin{aligned}
&\tilde{q}_j\big(\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big) \\
&= \sum_{\boldsymbol{x}(:,j)} f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \cdot \mathrm{e}^{\langle \boldsymbol{R}(:,j), \boldsymbol{x}(:,j)\rangle} \cdot \big(\boldsymbol{\beta}_{\mathcal{E}}(:,j)\big)^{\mathbf{1}-\boldsymbol{x}(:,j)}
\end{aligned}
$$

is a real stable polynomial with respect to $\boldsymbol{\beta}_{\mathcal{E}}(:,j) \in \mathbb{C}^{n_j}$. Also, we have

$$
f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \cdot \mathrm{e}^{\langle \boldsymbol{R}(:,j), \boldsymbol{x}(:,j)\rangle} \in \mathbb{R}_{>0}, \qquad \boldsymbol{x}(:,j) \in \mathcal{X}_{f_{\mathrm{r},j}}.
$$

By Lemma 29, we know that

$$
\log\Big(\tilde{q}_j\big(\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big)\Big)
$$

is a concave function with respect to $\boldsymbol{\beta}_{\mathcal{E}}(:,j) \in [0,1]^{n_j}$. Because $\mathbf{1}-\boldsymbol{\beta}_{\mathcal{E}}(:,j)$ is an affine transformation and the composition of a concave function and an affine function is a again a concave function (see, *e.g.*, [30, Section 3]), we know that

$$
\log\Big(\tilde{q}_j\big(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big)\Big)
$$

is also concave with respect to $\boldsymbol{\beta}_{\mathcal{E}}(:,j) \in [0,1]^{n_j}$ as well.

The proof of the concavity of

$$
\log\Big(\tilde{p}_i\big(\boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:)\big)\Big)
$$

with respect to $\boldsymbol{\beta}_{\mathcal{E}}(i,:) \in [0,1]^{m_i}$ for each $i \in \mathcal{I}$ is actually easier and thus it is omitted here. ∎

Now we prove the statements in Items 1 and 2 in Theorem 15.

- The convexity of $\tilde{F}_{\mathrm{B,d}}$ with respect to $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$ for fixed $\boldsymbol{L}, \boldsymbol{R} \in \mathbb{R}^{|\mathcal{E}|}$ follows from Lemma 41.
- Fix $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$. For each $i \in \mathcal{I}$, the function
$$
-\log\Big(\tilde{p}_i\big(\boldsymbol{\beta}_{\mathcal{E}}(i,:), \boldsymbol{L}(i,:)\big)\Big)
$$

is the negative of a "log-sum-exponential" function with respect to $\boldsymbol{L}(i,:) \in \mathbb{R}^{m_i}$, which is a concave function. Similarly, for each $j \in \mathcal{J}$, the function

$$-\log\Big(\tilde{q}_j\big(\boldsymbol{\beta}_{\mathcal{E}}(:,j), \boldsymbol{R}(:,j)\big)\Big)$$

is a concave function with respect to $\boldsymbol{R}(:,j) \in \mathbb{R}^{n_j}$. Therefore, the function $\tilde{F}_{\mathrm{B,d}}$ is a concave function with respect to $\boldsymbol{L}, \boldsymbol{R} \in \mathbb{R}^{|\mathcal{E}|}$ for fixed $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$.

Appendix F
Proof of Corollary 16

The definition of $\mathcal{L}_{\mathcal{E}}$ in Definition 6 implies that the set $\mathcal{L}_{\mathcal{E}}$ is convex. Thus to prove that the minimization problem in (1) is a convex optimization problem, it is sufficient to prove the convexity of $F_{\mathrm{B},\mathcal{E}}$ for $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$.

Following the similar idea in the proof of Theorem 14, we can use the Lagrangian dual method to get the following equality:

$$F_{\mathrm{B},\mathcal{E}}(\boldsymbol{\beta}_{\mathcal{E}}) = \sup_{\boldsymbol{L},\boldsymbol{R}\in\mathbb{R}^{|\mathcal{E}|}} \tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}, \boldsymbol{R}\big), \qquad \boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}.$$

By Theorem 15, we know that $\tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}, \boldsymbol{R}\big)$ is a convex function with respect to $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$ for fixed $\boldsymbol{L}, \boldsymbol{R} \in \mathbb{R}^{|\mathcal{E}|}$. Then the supremum

$$\sup_{\boldsymbol{L},\boldsymbol{R}\in\mathbb{R}^{|\mathcal{E}|}} \tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}, \boldsymbol{R}\big)$$

is again a convex function with respect to $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$ (see, *e.g.*, [30, Section 3]). Therefore, $F_{\mathrm{B},\mathcal{E}}(\boldsymbol{\beta}_{\mathcal{E}})$ is a convex function with respect to $\boldsymbol{\beta}_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$.

Appendix G
Proof of Theorem 22

We prove a lemma first.

Lemma 42. *If the inverse likelihood ratios that constitute an SPA fixed point are all positive-valued, then the collection of beliefs $\boldsymbol{\beta}$ evaluated at this SPA fixed point, as defined in Definition 19, satisfies $\boldsymbol{\beta} \in \mathcal{L}$.*

Proof. This follows from the definition of the SPA fixed point in Definition 19 and the details are omitted here. ∎

Remark 43. *In this appendix, we prove that if an SPA fixed point consists of positive-valued inverse likelihood ratios only, then this fixed point corresponds to a stationary point of the function $\tilde{F}_{\mathrm{B,d}}$, which is the objective function in the dual formulation of the Bethe partition function, as defined in Definition 13. This is different from the proof in [6, Theorem 2], where they showed that such an SPA fixed point corresponds to a stationary point of the Bethe free energy function F_{B}, which is the objective function in the primal formulation of the Bethe partition function.*

Following Definition 19, given a collection of positive-valued inverse likelihood ratios $(\overleftarrow{\boldsymbol{V}}, \overrightarrow{\boldsymbol{V}})$ that constitutes an SPA fixed point, the collection of beliefs defined by

$$\boldsymbol{\beta}' \triangleq f_{\mathrm{SPA,b}}(\overleftarrow{\boldsymbol{V}}) \tag{8}$$

has positive-valued entries only. In the following, we show that $\boldsymbol{\beta}'$ is at the minimum of the Bethe free energy function F_{B}. By Lemma 42, we know that $\boldsymbol{\beta}' \in \mathcal{L}$ and the collection of beliefs of the edges $\boldsymbol{\beta}'_{\mathcal{E}}$ satisfies $\boldsymbol{\beta}'_{\mathcal{E}} = \boldsymbol{\beta}'_{\mathcal{I}} = \boldsymbol{\beta}'_{\mathcal{J}}$, which further implies $\boldsymbol{\beta}'_{\mathcal{E}} \in \mathcal{L}_{\mathcal{E}}$ and $0 < \beta'_{\mathcal{E}}(i,j) < 1$ for $(i,j) \in \mathcal{E}$. For each $(i,j) \in \mathcal{E}$, we define

$$L'(i,j) \triangleq \log\left(\frac{\overleftarrow{V}(i,j)}{\beta'_{\mathcal{E}}(i,j)}\right), \tag{9}$$

$$R'(i,j) \triangleq \log\Big(\overrightarrow{V}(i,j) \cdot \big(1 - \beta'_{\mathcal{E}}(i,j)\big)\Big). \tag{10}$$

Observing the equations of the belief evaluated at the SPA fixed point in Definition 19, for each $(i,j) \in \mathcal{E}$, we get

$$\exp\big(L'(i,j)\big) = \frac{1 + \overleftarrow{V}(i,j) \cdot \overrightarrow{V}(i,j)}{\overrightarrow{V}(i,j)},$$

$$\exp\big(R'(i,j)\big) = \frac{\overrightarrow{V}(i,j)}{1 + \overleftarrow{V}(i,j) \cdot \overrightarrow{V}(i,j)}.$$

Therefore, combining the property $\boldsymbol{\beta}' \in \mathcal{L}$ with the definition of $\mathcal{L}$ in Definition 6, for each $(i,j) \in \mathcal{E}$, we have

$$\frac{\partial}{\partial L(i,j)} \tilde{F}_{\mathrm{B,d}} = \frac{\partial}{\partial R(i,j)} \tilde{F}_{\mathrm{B,d}} = \frac{\partial}{\partial \beta_{\mathcal{E}}(i,j)} \tilde{F}_{\mathrm{B,d}} = 0$$

for $\boldsymbol{L} = \boldsymbol{L}'$, $\boldsymbol{R} = \boldsymbol{R}'$, and $\boldsymbol{\beta} = \boldsymbol{\beta}'$. By the convexity-concavity of $\tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}, \boldsymbol{R}\big)$, as proven Theorem 15, we know that

$$\begin{aligned} \sup_{\boldsymbol{L},\boldsymbol{R}\in\mathbb{R}^{|\mathcal{E}|}} \tilde{F}_{\mathrm{B,d}}(\boldsymbol{\beta}'_{\mathcal{E}}, \boldsymbol{L}, \boldsymbol{R}) &\leq \tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}'_{\mathcal{E}}, \boldsymbol{L}', \boldsymbol{R}'\big) \\ &\leq \min_{\boldsymbol{\beta}_{\mathcal{E}}\in\mathcal{L}_{\mathcal{E}}} \tilde{F}_{\mathrm{B,d}}(\boldsymbol{\beta}_{\mathcal{E}}, \boldsymbol{L}', \boldsymbol{R}'). \end{aligned} \tag{11}$$

Therefore, we get

$$\begin{aligned} Z_{\mathrm{B}}(\mathsf{N}) &\overset{(a)}{=} \tilde{Z}_{\mathrm{B,d}}(\mathsf{N}) \\ &\overset{(b)}{=} \exp\Big(-\tilde{F}_{\mathrm{B,d}}\big(\boldsymbol{\beta}'_{\mathcal{E}}, \boldsymbol{L}', \boldsymbol{R}'\big)\Big) \\ &\overset{(c)}{=} \exp\big(-F_{\mathrm{B},\#}(\overleftarrow{\boldsymbol{V}}, \overrightarrow{\boldsymbol{V}})\big) \\ &\overset{(d)}{=} \exp(-F_{\mathrm{B}}(\boldsymbol{\beta}')), \end{aligned}$$

- where step (a) follows from Theorem 14,
- where step (b) follows from the definition of $\tilde{Z}_{\mathrm{B,d}}$ in Definition 13 and the inequalities in (11),
- where step (c) follows from the expressions for the entries in $\boldsymbol{L}'$ and $\boldsymbol{R}'$ in (9) and (10), the expression in (8), and the expression of $F_{\mathrm{B},\#}$ in Lemma 20,
- where step (d) follows from the expression in (8) and the definition of F_{B} in Definition 9.

Appendix H
Proof of Theorem 23

The main idea of the proof is as follows.

1) We first observe that the SPA update rules with respect to the inverse likelihood ratios in Lemma 18 can be rewritten in terms of the partial derivatives of the polynomials p_i and q_j.
2) Because both p_i and q_j are MAHRS polynomials, we apply [22, Theorem 5.6] to show that the partial derivatives of these polynomials, *i.e.*, the SPA update rules, have some monotonic properties.
3) Based on these properties, we prove the exponential convergence of the SPA following similar ideas as in [11, Appendix G-B].

We first show some monotonic property in the message update rules.

$$0 = \left(1 + \overleftarrow{V}(i,j) \cdot \overrightarrow{V}(i,j)\right)^{-1} \cdot \left(p_i\big(\overleftarrow{\boldsymbol{V}}(i,:)\big)\right)^{-1} \cdot \left(\sum_{\boldsymbol{x}(i,:):\, x(i,j)=0} f_{1,i}\big(\boldsymbol{x}(i,:)\big) \cdot \big(\overleftarrow{\boldsymbol{V}}(i,:)\big)^{\boldsymbol{x}(i,:)} \right)$$
$$\cdot \left(\overrightarrow{V}(i,j) - \frac{\sum\limits_{\boldsymbol{x}(i,:):\, x(i,j)=1} f_{1,i}\big(\boldsymbol{x}(i,:)\big) \cdot \prod\limits_{j_1 \in \mathcal{J}_i \setminus \{j\}} \left(\overleftarrow{V}(i,j_1)\right)^{x(i,j_1)}}{\sum\limits_{\boldsymbol{x}(i,:):\, x(i,j)=0} f_{1,i}\big(\boldsymbol{x}(i,:)\big) \cdot \big(\overleftarrow{\boldsymbol{V}}(i,:)\big)^{\boldsymbol{x}(i,:)}} \right). \tag{12}$$

$$\frac{\partial}{\partial \overleftarrow{V}_{t-1}(i,j_1)} \overrightarrow{V}_t(i,j) = \left(p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)\right)^{-2} \cdot \Bigg(p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big) \cdot \frac{\partial^2}{\partial \overleftarrow{V}_{t-1}(i,j)\, \partial \overleftarrow{V}_{t-1}(i,j_1)} p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)$$
$$- \frac{\partial}{\partial \overleftarrow{V}_{t-1}(i,j)} p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big) \cdot \frac{\partial}{\partial \overleftarrow{V}_{t-1}(i,j_1)} p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big) \Bigg) \Bigg|_{\overleftarrow{V}_{t-1}(i,j)=0}$$
$$\overset{(a)}{\leq} 0. \tag{13}$$

Lemma 44. *For each $i \in \mathcal{I}$, $j \in \mathcal{J}$, and $t \in \mathbb{Z}_{\geq 1}$, we rewrite the SPA update rules in Lemma 18 in terms of $p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)$ and $q_j\big(\overrightarrow{\boldsymbol{V}}_t(:,j)\big)$ as follows*

$$\overrightarrow{V}_t(i,j) = \frac{\partial}{\partial \overleftarrow{V}_{t-1}(i,j)} \log\Big(p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)\Big)\Big|_{\overleftarrow{V}_{t-1}(i,j)=0},$$
$$\overleftarrow{V}_t(i,j) = \frac{\partial}{\partial \overrightarrow{V}_t(i,j)} \log\Big(q_j\big(\overrightarrow{\boldsymbol{V}}_t(:,j)\big)\Big)\Big|_{\overrightarrow{V}_t(i,j)=0}.$$

Proof. This follows from the definitions of $p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)$ and $q_j\big(\overrightarrow{\boldsymbol{V}}_t(:,j)\big)$ in Definition 3 and the SPA update rules in Lemma 18. ∎

Remark 45. *There is another perspective to understand Lemma 44 from $F_{\mathrm{B},\#}$. Consider positive-valued inverse likelihood ratios $(\overleftarrow{\boldsymbol{V}}', \overrightarrow{\boldsymbol{V}}')$.*

For each $(i,j) \in \mathcal{E}$, one of the equations for the stationary point of $F_{\mathrm{B},\#}$ is given in (12) at the top of this page. Let us focus on expression on the RHS of the equality in (12). We note that the first three terms are positive-valued for $(\overleftarrow{\boldsymbol{V}}, \overrightarrow{\boldsymbol{V}}) = (\overleftarrow{\boldsymbol{V}}', \overrightarrow{\boldsymbol{V}}')$. Thus the equality in (12) holds for $(\overleftarrow{\boldsymbol{V}}, \overrightarrow{\boldsymbol{V}}) = (\overleftarrow{\boldsymbol{V}}', \overrightarrow{\boldsymbol{V}}')$ if and only if the forth term is zero. Similar properties also hold for other equations for the stationary point of $F_{\mathrm{B},\#}$. After solving these stationary-point equations, we get the equations for $(\overleftarrow{\boldsymbol{V}}', \overrightarrow{\boldsymbol{V}}')$ being at the fixed point for the SPA update rules in Lemma 18.

Now we consider $(\overleftarrow{\boldsymbol{V}}', \overrightarrow{\boldsymbol{V}}')$ being at an SPA fixed point. If we set

$$\overleftarrow{V}(i,j_1) = \overleftarrow{V}'(i,j_1), \qquad j_1 \in \mathcal{J}_i \setminus \{j\},$$
$$\overleftarrow{V}(i,j) = 0,$$

then the forth term on the RHS of the equality in (12) still equals zero, i.e., the equality in (12) still holds. In this case, the equality in (12) corresponds to the equation for $\overrightarrow{V}'(i,j)$ being at the fixed point of the SPA update rules in Lemma 44. Similar properties also hold for other equations for the stationary point of $F_{\mathrm{B},\#}$. After solving the stationary-point equations under this setup, we get the fixed-point equations for the SPA update rules in Lemma 44.

Lemma 46. *For each $(i,j) \in \mathcal{E}$ such that $\overrightarrow{V}_t(i,j) \in \mathbb{R}_{>0}$, the ratio $\overrightarrow{V}_t(i,j)$ is a non-increasing function with respect to $\overleftarrow{V}_{t-1}(i,j_1) \in \mathbb{R}_{\geq 0}$ for all $j_1 \in \mathcal{J}_i \setminus \{j\}$. Similarly, for $\overleftarrow{V}_t(i,j) \in \mathbb{R}_{>0}$, the ratio $\overleftarrow{V}_t(i,j)$ is a non-increasing function with respect to $\overrightarrow{V}_t(i_1,j) \in \mathbb{R}_{\geq 0}$ for all $i_1 \in \mathcal{I}_j \setminus \{i\}$.*

Proof. The proof of the second statement in the lemma is similar to the proof of the first statement in the lemma. Therefore, it is sufficient to prove the the first statement.

Consider an arbitrary $j_1 \in \mathcal{J}_i \setminus \{j\}$. We obtain

$$\overrightarrow{V}_t(i,j)$$
$$\overset{(a)}{=} \frac{1}{p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)} \cdot \frac{\partial}{\partial \overleftarrow{V}_{t-1}(i,j)} p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)\Big|_{\overleftarrow{V}_{t-1}(i,j)=0}$$
$$\overset{(b)}{>} 0, \tag{14}$$

where step (a) follows from the SPA update rule in Lemma 44, and where step (b) follows from $\overrightarrow{V}_t(i,j) \in \mathbb{R}_{>0}$ as stated in the lemma statement. By the inequality in (14) and the fact that for all $\overleftarrow{\boldsymbol{V}}_{t-1}(i,:) \in \mathbb{R}_{\geq 0}^{m_i}$, the following inequalities hold:

$$\frac{\partial}{\partial \overleftarrow{V}_{t-1}(i,j)} p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)\Big|_{\overleftarrow{V}_{t-1}(i,j)=0} > 0,$$
$$p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)\Big|_{\overleftarrow{V}_{t-1}(i,j)=0} > 0. \tag{15}$$

The partial derivative of $\overrightarrow{V}_t(i,j)$ with respect to $\overleftarrow{V}_{t-1}(i,j_1)$ is given in (13) near the top of this page, where step (a) follows from the strict inequality in (15), from Theorem 30, and the fact that $p_i\big(\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)\big)$ is an MAHRS polynomial with respect to $\overleftarrow{\boldsymbol{V}}_{t-1}(i,:)$, as defined in Definition 3. ∎

Based on the monotonic proven in Lemma 46, we can use similar ideas as in [11, Appendix G-B] to prove Theorem 23. The details are omitted here.