# IERG4150 Intro. to Cryptography

Sherman Chow
Chinese University of Hong Kong
Fall 2024
Lecture 1: One-time Pad and its Security Proof

# Fundamentals of "Provable Security"

- Security: It is a nebulous concept, but not if you took this course

- Provable:
  - We can formally define what it means to be secure
  - and then mathematically prove claims about security
  - *e.g.*, logic of composing building blocks together in secure ways

- Fundamentals:
  - solid theoretical foundation applicable to most real-world situations
  - equipped to (self-)study more advanced topics in cryptography

# What (Modern) Cryptography is?

- not a magic spell that solves all security problems
- providing solutions to *cleanly defined* problems
  - often *abstract away* important but messy real-world concerns
- "Cryptographic guarantees"/"Provable security":
  - What happens (or what cannot happen) in the presence of certain well-defined classes of attacks
  - What if the model is too restrictive (in defining the attacks)?
  - What if the "real-world" attackers don't follow the "rules"?
  - Disappointing/Underwhelming?

# Defining Security

- Making the nebulous concept of "security" concrete
- Breaking the vicious circle of "cat-and-mouse" games

- We will try to model the attacker as "powerful" as possible

- Always keep in mind: we define (*i.e.*, limit) our problems

*"To define is to limit."*
*—Oscar Wilde*

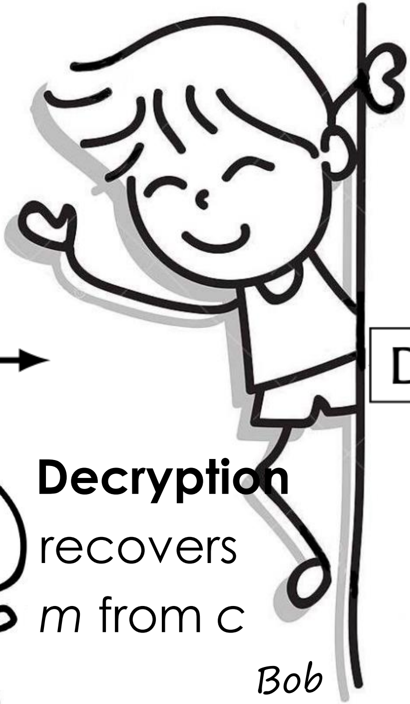
# "Private" (Confidential) Communication


- **Plaintext**: *m*
- **Ciphertext**: c
- **Encryption** turns *m* into c

$$m \to \boxed{\mathrm{Enc}} \xrightarrow{c} \boxed{\mathrm{Dec}} \to m$$

- **Eave**sdropper can (passively) observe the communication


**Decryption** recovers *m* from c



# Secret, or secrecy of the algorithms?

- We want Bob to be able to decrypt c
- but Eve to not be able to decrypt c
- Suppose Eve has unbounded computational power
- [Exercise] Argue that both sender and receiver must share a secret not known to the adversary
- Hide the details of the Enc() and Dec() algorithms secret?
  - how crypto was done throughout most of the last 2000 years
  - but it has major drawbacks!

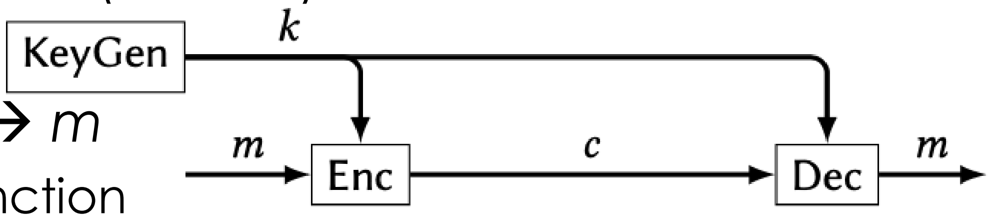
# Kerckhoffs' Principle

*"Il faut qu'il n'exige pas le secret, et qu'il puisse sans inconvénient tomber entre les mains de l'ennemi."*

- A system designer wants the system to be widely used.
- It is hard to keep a secret (*e.g.*, reverse engineering).
- If details of Enc() and Dec() are leaked, what can we do?
- Invent a new encryption system!
  - Inventing even a good one is already hard enough!
- [The method] must not be required to be secret, and it must be able to fall into the enemy's hands without causing inconvenience.
- Bottom line: Design your system to be secure even if the attacker has complete knowledge of all its algorithms.
  - vs. security by obscurity

# What constitutes an encryption scheme?

- Key generation algorithm (KeyGen)
  - Input: security parameter $\lambda$ (lambda)
  - Output: a key $k$
- $\mathrm{Enc}_k(m) \rightarrow c$, $\mathrm{Dec}_k(c) \rightarrow m$
  - *i.e.*, they are key-ed function
  - All these algorithms are supposed to be public




- A crypto scheme/construction is a collection of algorithms
- *Symmetric*-key encryption = (KeyGen, Enc, Dec)

# Syntax forms the basis of Security

- We call the inputs/outputs (*i.e.*, the "function signature") of the various algorithms the *syntax* of the scheme.

- KeyGen is a *probabilistic/randomized* algorithm
- Knowing the details (*i.e.*, source code) of a randomized algorithm does *not* mean you know the *specific* output it gave when it was executed

- Encoding/decoding methods is *not* encryption [Why?]
  - What is "b25seSBuZXJkcyB3aWxsIHJlYWQgdGhpcw=="?

# What are outside our model's protection?

- The fact that Alice is sending something to Bob
  - We only want to hide the *contents* of that message
  - Steganography hides the existence of a communication channel
- How c reliably gets from Alice to Bob
- We aren't considering an attacker that tampers with c (causing Bob to receive and decrypt a different value)
  - We will consider such attacks later though

# What it takes in the "real world"?

- How Alice and Bob actually obtain a common secret key
- How they can keep them secret while (keep) using it
- How to uniformly sample random (bit-)strings?
  - No randomness, no cryptography
  - Obtaining uniformly random bits from *deterministic* computers is extremely non-trivial

"Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin."
— John von Neumann

# Probabilistic Polynomial Time (PPT) Algo.

- $y = A(x)$
- Input x is of *size/length n*
  - We write $|x| = n$
- A PPT algorithm has $O(n^c)$ run-time, c being a constant
  - We say a PPT algorithm is an "efficient" algorithm
- Probabilistic: allows "flipping a coin" to make it randomized
- $y \leftarrow A(x)$
- $y$ denotes the random variable corresponds to $A$'s output
- Or $y = A(x; r)$, where $r$ denotes $A$'s "*coin tossing*"
  - $r$'s length is also polynomial in $n$
  - when we had the need to specify the randomness explicitly

# Negligible Function

- A function $\nu(n)$ is called negligible, denoted **negl**($n$), if:
- $(\forall c > 0)\ (\exists n')\ (\forall n \geq n')\ [|\nu(n)| \leq 1/n^c]$
- Less than the inverse of any polynomial for large enough $n$

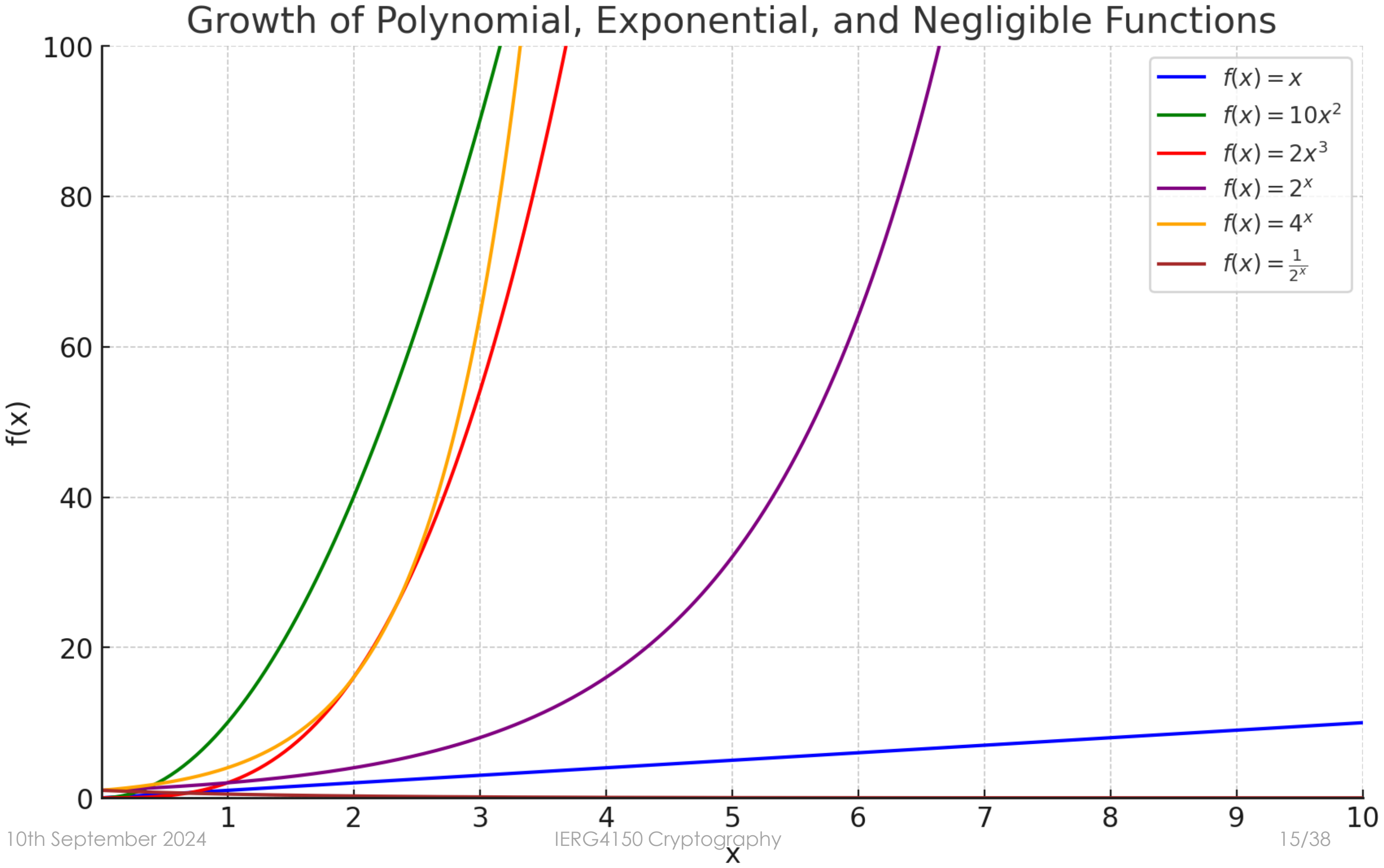
$\leq 1/n^c$ $\forall c > 0$ $(\forall n \geq n')$

- Prob. of breaking a secure system should be negligible in $n$
  - a practically zero value (for sufficiently large inputs)

- Let **poly**($n$) denote some polynomial function in $n$
- We have poly($n$) · negl($n$) = negl($n$) (abusing notations)

# Explaining Negligible Function [*]

- $(\forall c > 0)\ (\exists n')\ (\forall n \geq n')\ [|v(n)| \leq 1/n^c]$
- For all $c > 0$:
  - "Pick any speed you want, and I'll prove to you that this function shrinks even faster than that."
  - $c$ controls how fast we want the function to shrink.
  - The bigger $c$ is, the faster we're asking $v(n)$ to shrink as $n$ gets larger.
- There exists $n'$, for all $n \geq n'$
  - "Before $n'$, we don't care much about the function's behavior. We're only concerned with what happens when $n'$ becomes large."
  - $n'$ is just a starting point, after which $v(n)$ behaves in a certain way.
- for all $n \geq n'$, $|v(n)| \leq 1/n^c$:
  - No matter how small or fast you make this fraction by choosing a large $c$, $v(n)$ can't be bigger than that fraction once $n$ is big enough.
  - The larger $c$ is, faster $1/n^c$ becomes small, so $v(n)$ must shrink even faster


Growth of Polynomial, Exponential, and Negligible Functions
$f(x) = x$
$f(x) = 10x^2$
$f(x) = 2x^3$
$f(x) = 2^x$
$f(x) = 4^x$
$f(x) = \frac{1}{2^x}$
f(x)
x
0
20
40
60
80
100
1
2
3
4
5
6
7
8
9
10

# Security Parameter (& some notations)

- We want a "set" of cryptosystems parameterized by $n$
- Algo.'s run by all parties take commonly agreed input $n$
- They run in time *polynomial* in their input length $n$
- Summary of Notations:
  - poly($n$): runtime of all parties are sufficiently fast, *e.g.*, $n^3$
  - negl($n$): *e.g.*, $1/2^n$ is in negl($n$)
  - $\{0, 1\}^n$ : the set of $n$ symbols, where each of the $n$ symbols is 0 or 1
  - $1^n$ (unlike $2^n$ above) is a string with $n$ "copies" of 1's, *i.e.*, $1^n$ is in $\{0, 1\}^n$
- Security parameter of the system is $1^n$ (with length $n$ bits)
  - If we put $n$ as an input, the length of (input) $n$ is log($n$) bits

# Tasks of Crypto. Study ([*] / [**])

- Identification of the problem / application scenario
- Identification of the primitive which may be useful
  - Do not re-invent the wheel
  - Extending existing primitives
  - Relation between primitives (one implies another?)
- Definition of Functional Requirements
  - A suite of algorithms / protocols, their input & output behavior / interfaces
  - System model: what entities are involved, which entity executes which algorithm/protocols
- Definition of Security requirements
  - Relation of security notions (one implies another?)
- Construction of the schemes
- Analysis of the proposed construction
  - Security Proof: Provable Security!
  - Efficiency (Order Analysis and/or Experiment on Prototype Implementation)

**Notation in the Slides**
[*]: slightly complicated,
slides did not give full details,
but it should make sense to you.
[**]: advanced materials,
not much details provided,
"out-of-syllabus"

# Attackers' Goal vs. Strength of Encryption

- Recover the plaintext *m*
- Recover a part of the plaintext *m*
  - (Weaker adversary)
  - To protect against a weaker adversary, a weaker scheme may suffice
  - The weaker scheme might be more efficient
- Recover the secret key
  - (Stronger adversary)

- "Deem" only a break when…

- Whole *m* is recovered
  - (Weakest security level)

- Some part of *m* is recovered
  - (Slightly stronger)

- "1 bit information" of *m* is leaked
  - (Strongest)
  - May not be the actual bit of *m*
    - Consider *m* is known to be "yes" or "no"

# One-Time Pad (OTP) based on XOR

- eXclusive OR (XOR): For $b_1 \oplus b_2$ ($b_i$ is a bit), output as the table
  - a logical operator that returns 1 (true) if the number of 1 (true) inputs is odd
  - XOR is also addition modulo 2 (1 + 1 = 2, 2 mod 2 = 0)
- For bit-string operation $S_1 \oplus S_2$, just $\oplus$ in a bit-wise manner
- OTP = {KeyGen, Enc, Dec}
- KeyGen($1^\lambda$):
  - *uniformly* sample a λ-bit string *k*
  - output *k*
- Enc($k$, $m$) → $c = m \oplus k$;
  - ($m$ is λ-bit long)
- Dec($k$, $c$) → $m = c \oplus k$

$b_2$

| XOR $\oplus$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

$b_1$

$$\underline{\textbf{KeyGen:}}$$
$$k \leftarrow \{0,1\}^\lambda$$
$$\textbf{return } k$$

$$\underline{\text{Enc}(k, m \in \{0,1\}^\lambda):}$$
$$\textbf{return } k \oplus m$$

$$\underline{\text{Dec}(k, c \in \{0,1\}^\lambda):}$$
$$\textbf{return } k \oplus c$$

# Example

- OTP-encrypt the 20-bit plaintext *m* below under a key *k*:

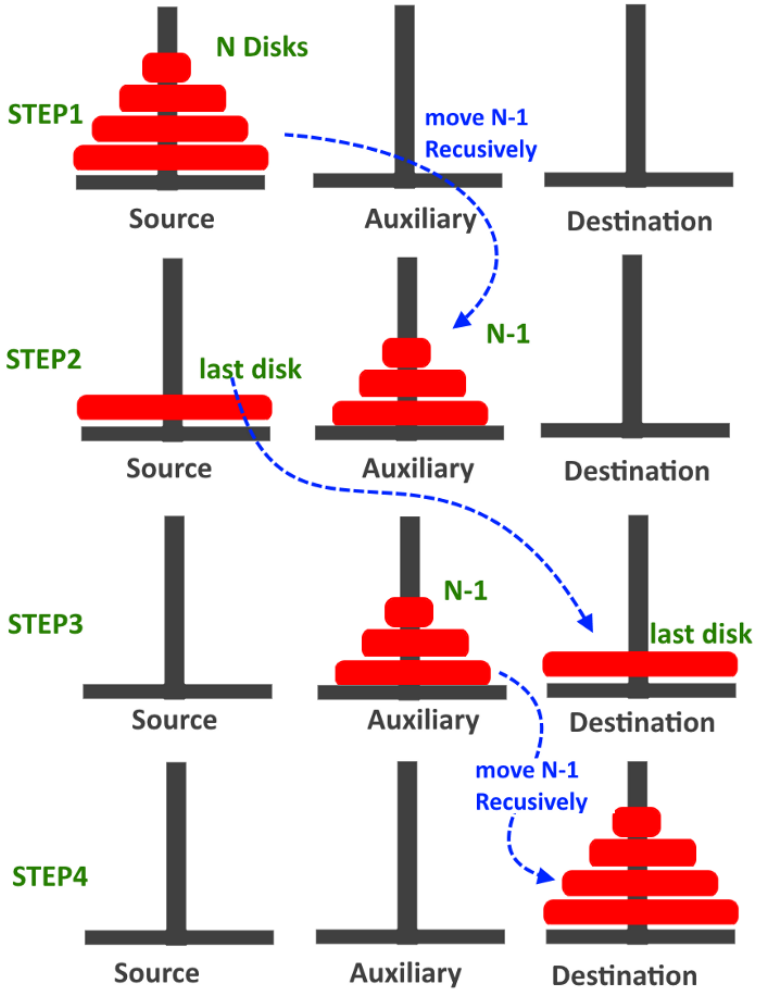
$$
\begin{array}{rll}
 & \texttt{11101111101111100011} & (m) \\
\oplus & \texttt{00011001110000111101} & (k) \\
\hline
 & \texttt{11110110011111011110} & (c = \mathsf{Enc}(k, m))
\end{array}
$$

- OTP-decrypt the 20-bit ciphertext *c* below under a key *k*:

$$
\begin{array}{rll}
 & \texttt{00001001011110010000} & (c) \\
\oplus & \texttt{10010011101011100010} & (k) \\
\hline
 & \texttt{10011010110101110010} & (m = \mathsf{Dec}(k, c))
\end{array}
$$

# Detour: Algorithms

- I could dry-run an algorithm with concrete examples if I were teaching an algorithm course
  - Not exactly concrete details ->
  - "Abstracted away" by the recursive calls
  - but I could flatten it out if I want to
- How about crypto algorithms?




Credit: https://medium.com/@jamalmaria111/tower-of-hanoi-js-algorithm-3f667fa46f0f

# Crypto. Algorithms

- I have provided concrete examples (don't say I didn't :), but, what did you learn by these examples?

$$
\begin{array}{rll}
 & \texttt{111011111011111100011} & (m) \\
\oplus & \texttt{000110011100001111101} & (k) \\
\hline
 & \texttt{111101100111111011110} & (c = \mathsf{Enc}(k, m))
\end{array}
$$

- You saw how Enc() (or Dec()) works for a particular input
- You get a *sense* of correctness ($m$ = Dec($k$, Enc($k$, $m$)))
- But how can you argue about its security?

# Why Cryptography is difficult?

- Security is a **global** property about the behavior of a system across **all possible** inputs.
  - You can't demonstrate security by example,
  - and there's nothing to see in a particular execution of an algorithm.
- Security is about a **higher level of abstraction**.
  - (and some students might not be comfortable with it)
- Most security definitions in this course are essentially:
  - "*the thing is secure if its outputs look like random junk.*"
  - *i.e.*, any example just look like meaningless garbage

# Correctness of OTP

- For all $k, m \in \{0, 1\}^\lambda$, it is true that $\mathrm{Dec}(k, \mathrm{Enc}(k, m)) = m$.
- More precisely: For all $m$ in the *message space* $\mathbf{M} = \{0, 1\}^\lambda$ and all $k$ in the *key space* $\mathbf{K} = \{0, 1\}^\lambda$, it is true that $\mathrm{Dec}(k, \mathrm{Enc}(k, m)) = m$.
- Or simply, one can *always* recover $m$.
- Proof:
- $\quad \mathrm{Dec}(k, \mathrm{Enc}(k, m))$
- $= \mathrm{Dec}(k, k \oplus m)$
- $= k \oplus (k \oplus m)$
- $= (k \oplus k) \oplus m$ // $\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- $= 0^\lambda \oplus m = m$.

| XOR $\oplus$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

# Cautions: OTP is unique in its own ways

- (patented in 1919, but recently discovered in an 1882 text)
- The security crucially depends on sampling $k$ ***uniformly*** at random from the set of $\lambda$-bit strings
  - The security would not hold if it is under **other distribution**.
- (This step in) KeyGen() is the only source of randomness
  - we'll see using randomness "more" (*e.g.*, in more algorithms) later
- Enc() and Dec() are "essentially" the same algorithm
  - but it is more of a coincidence than something truly fundamental
- Message space, key space, are just the ciphertext space
  - again, just a special case here, nothing is said in general

# Security Proof

- *"Because of the <u>specific</u> way the ciphertext was generated, it doesn't reveal any information about the plaintext to the attacker, no matter what the attacker does with the ciphertext."*

- We need to first specify how the ciphertext is generated.
- Didn't we? It is the encryption algorithm
  - (which relies on KeyGen())
- But it was from the point of view of "honest" users Alice and Bob
- How can I predict "what the attacker does with the ciphertext"?
  - Yes, but at least we need to specify what ciphertext does *it* see.

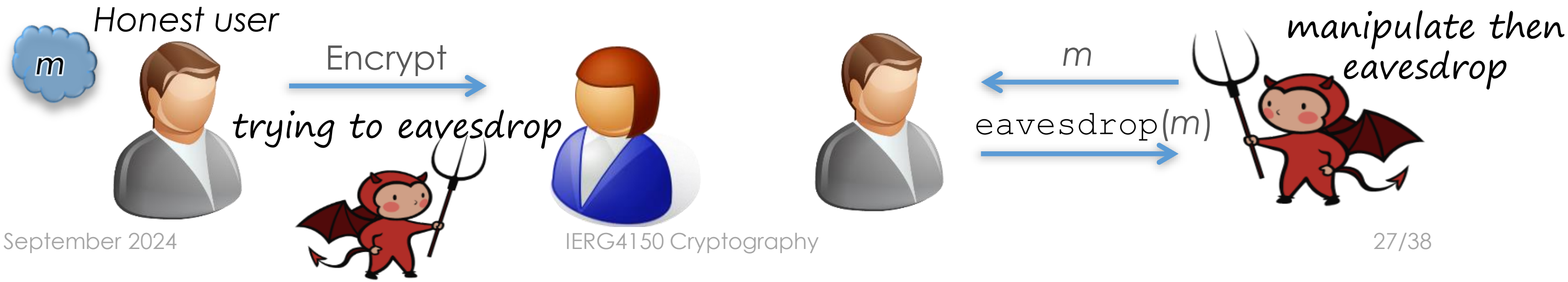
# Modelling what the adversary sees

- We always treat the attacker as some (unspecified) process that receives output from an algorithm (`eavesdrop` here).
- not what the attacker does
- but rather the process

  (carried out by honest users)

  that produces what the attacker sees

$$\underline{\textsc{eavesdrop}(m \in \{0,1\}^\lambda):}$$
$$k \leftarrow \{0,1\}^\lambda$$
$$c := k \oplus m$$
$$\text{return } c$$



# Probabilistic Alg. & its Output Distribution

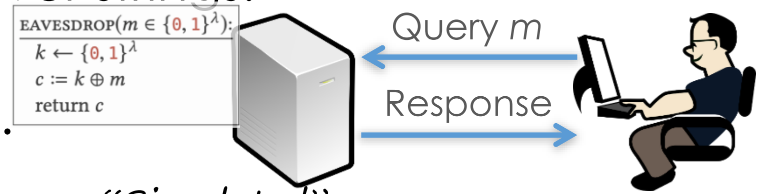
- Our goal: "*the output of* `eavesdrop` *doesn't reveal the input m.*"

- If you call `eavesdrop` several times,
- even on the same input,
- you are likely to get different outputs.

$$\begin{array}{l}\underline{\text{EAVESDROP}(m \in \{0,1\}^\lambda):}\\ \quad k \leftarrow \{0,1\}^\lambda \\ \quad c := k \oplus m \\ \quad \text{return } c\end{array}$$

- Instead of thinking of "`eavesdrop`$(m)$" as a single string,
- think of it as a *probability distribution* over strings.
- Each time you call `eavesdrop`$(m)$,
- you see a *sample* from the distribution.

Attacker algorithm




"Simulated" honest user

# (Toy) Example

- $\lambda$ = 3 and consider `eavesdrop`(010) and `eavesdrop`(111).

EAVESDROP(010):

| Pr | $k$ | output $c = k \oplus 010$ |
|---|---|---|
| 1/8 | 000 | 010 |
| 1/8 | 001 | 011 |
| 1/8 | 010 | 000 |
| 1/8 | 011 | 001 |
| 1/8 | 100 | 110 |
| 1/8 | 101 | 111 |
| 1/8 | 110 | 100 |
| 1/8 | 111 | 101 |

EAVESDROP(111):

| Pr | $k$ | output $c = k \oplus 111$ |
|---|---|---|
| 1/8 | 000 | 111 |
| 1/8 | 001 | 110 |
| 1/8 | 010 | 101 |
| 1/8 | 011 | 100 |
| 1/8 | 100 | 011 |
| 1/8 | 101 | 010 |
| 1/8 | 110 | 001 |
| 1/8 | 111 | 000 |

every string in the ciphertext space ($\{0, 1\}^\lambda$) appears <u>exactly</u> once, with the same (1/8) probability

a. k. a. uniform distribution over $\{0, 1\}^\lambda$

k is chosen uniformly at random from $\{0, 1\}^\lambda$

# Some conclusions

- Nothing special about 010 or 111 in the above examples.
- The distribution `eavesdrop`($m$) is the uniform distribution over the ciphertext space $\{0, 1\}^\lambda$.
- Let's formalize this argument (without tabulating $2^3$ times).

- Let's first formalize what we want to prove:
- "For every $m \in \{0, 1\}^\lambda$, the distribution `eavesdrop`($m$) is the uniform distribution on $\{0, 1\}^\lambda$."
- Corollary: For every $m, m' \in \{0, 1\}^\lambda$, the distributions `eavesdrop`($m$) and `eavesdrop`($m'$) are identical.

# The Exact Proof from the Textbook

**Proof** Arbitrarily fix $m, c \in \{0,1\}^\lambda$. We will calculate the probability that $\textsc{eavesdrop}(m)$ produces output $c$. That event happens only when

$$c = k \oplus m \iff k = m \oplus c.$$

The equivalence follows from the properties of XOR given in Section 0.3. That is,

$$\Pr[\textsc{eavesdrop}(m) = c] = \Pr[k = m \oplus c],$$

where the probability is over uniform choice of $k \leftarrow \{0,1\}^\lambda$.

We are considering a specific choice for $m$ and $c$, so there is *only one* value of $k$ that makes $k = m \oplus c$ true (causes $m$ to encrypt to $c$), and that value is exactly $m \oplus c$. Since $k$ is chosen *uniformly* from $\{0,1\}^\lambda$, the probability of choosing the particular value $k = m \oplus c$ is $1/2^\lambda$.

In summary, for every $m$ and $c$, the probability that $\textsc{eavesdrop}(m)$ outputs $c$ is exactly $1/2^\lambda$. This means that the output of $\textsc{eavesdrop}(m)$, for any $m$, follows the uniform distribution. ■

# What did we prove? (Part I)

- "For every $m \in \{0, 1\}^\lambda$, the distribution `eavesdrop`($m$) is the uniform distribution on $\{0, 1\}^\lambda$"; or (in "English"):
  - *"If an attacker sees a single ciphertext,*
  - *encrypted with one-time pad, where the key*
  - *is chosen uniformly and kept secret from the attacker,*
  - *then the ciphertext appears uniformly distributed."*
- Suppose someone chooses a plaintext $m$.
- You (the attacker) get to see the resulting ciphertext —
- a sample from the distribution you can sample by yourself
- even if you don't know $m$!

# Security of OTP, and some discussions

- The "real" ciphertext doesn't carry *any information* about $m$ if it is possible to sample without even knowing $m$!

- Paradox 1: "One can *always* recover $m$ [from c]" contradicts with "c contains no information about $m$."
- The correctness proof assumes one w/ the knowledge of $k$
- Paradox 2: "`eavesdrop`($m$) does not depend on $m$" is blatantly false simply because it takes $m$ as an input!
- Our example shows that, when $m$ is different,

  the tabulated outputs indeed are different ($m$'s "effect")
- The claim is about they are being the same <u>distribution</u>.

# What did we prove? (Part II)

- For every $m$, $m' \in \{0, 1\}^{\lambda}$, the distributions `eavesdrop`($m$) and `eavesdrop`($m'$) are identical.
  - *"If an attacker sees a single ciphertext,*
  - *encrypted with one-time pad, where the key*
  - *is chosen uniformly and kept secret from the attacker,*
  - *for every two possibilities of the plaintext,*
  - *the resulting ciphertext appears from the same distribution"*
- The attacker's "view" is the same no matter what $m$ is
- and no matter what the plaintext distribution is!
  - (*cf.*, Caesar cipher…)

# What did we prove? (Part III)

- "For every $m \in \{0, 1\}^\lambda$, the distribution $\texttt{eavesdrop}(m)$ is the uniform distribution on $\{0, 1\}^\lambda$"

- Here, we consider some hypothetical "ideal" world:
- Any attacker sees only a source of uniform bits.
- There are no keys and no plaintexts to recover.

# What did we prove? (fin.)

- "For every $m \in \{0, 1\}^\lambda$, the distribution `eavesdrop`($m$) is the uniform distribution on $\{0, 1\}^\lambda$"

- Nothing was said about the attacker's goal!
  - *e.g.*, recovering the plaintext or the key
  - Looking ahead, we may do that in alternative definitions or cases
  - but we still want to be general enough
- What we prove: Any attacker, who saw an OTP ciphertext in the real world, has a point of view like in our hypothetical world!
- Or, it is a "modest" goal: detect that ciphertexts don't follow a uniform distribution (so harder goals are out of reach)

# Limitations of One-Time Pad

1. It can only be used once (to encrypt a single plaintext).

- Note that the `eavesdrop` procedure provides no way for a caller to guarantee that two calls will use the same key.
- So, we did not prove anything about reusing the key.

2. The key is as long as the plaintext
   - provably unavoidable (a.k.a. the key length is optimal) [*]

- Chicken-and-egg dilemma in practice:
  - If two users want to privately convey a $\lambda$-bit message,
  - they first need to privately agree on a $\lambda$-bit string.

# Then why teach OTP?

- Pedagogical: It illustrates fundamental **ideas** that appear in most forms of encryption in this course.
  - (recall the "Cautions" slide though)
- In "real-world": the *only* "*perfectly secure*" encryption scheme
  - imagine if someone sells a "perfect" encryption scheme to you…
- We propose the first solution, it may not be "ideal" (*e.g.*, inefficient)
- then we try to "twist" it to make it achieve some "better trade-offs"
  - How "innovation" work sometimes
- What if the attacker has bounded computation power?
- What if we manage to have some "pseudorandom strings"?
  - We'll study computationally-secure pseudo-random number generator