Learning-Aided Stochastic Network Optimization With State Prediction

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Abstract—We investigate the problem of stochastic network optimization in the presence of state prediction and non-stationarity. Based on a novel state prediction model featured with a distribution-accuracy curve, we develop the predictive learning-aided control (PLC) algorithm, which jointly utilizes historic and predicted network state information for decision making. PLC is an online algorithm that consists of three key components, namely, sequential distribution estimation and change detection, dual learning, and online queue-based control. We show that for stationary networks, PLC achieves a near-optimal utility-delay tradeoff. For non-stationary networks, PLC obtains an utility-backlog tradeoff for distributions that last longer than a time proportional to the square of the prediction error, which is smaller than that needed by backpressure (BP) for achieving the same utility performance. Moreover, PLC detects distribution change $O(w)$ slots faster with high probability ($w$ is the prediction size) and achieves a convergence time faster than that under BP. Our results demonstrate that state prediction helps: 1) achieve faster detection and convergence and 2) obtain better utility-delay tradeoffs. They also quantify the benefits of prediction in four important performance metrics, i.e., utility (efficiency), delay (quality-of-service), detection (robustness), and convergence (adaptability) and provide new insight for joint prediction, learning, and optimization in stochastic networks.

Index Terms—Network optimization, learning, prediction.

I. INTRODUCTION

ENABLED by recent developments in sensing, monitoring, and machine learning methods, utilizing prediction for performance improvement in networked systems has received a growing attention in both industry and research. For instance, recent research works [2]–[4] investigate the benefits of utilizing prediction in energy saving, job migration in cloud computing, and video streaming in cellular networks. On the industry side, various companies have implemented different ways to take advantage of prediction, e.g., Amazon utilizes prediction for better package delivery [5] and Facebook enables prefetching for faster webpage loading [6]. However, despite the continuing success in these attempts, most existing results in network control and analysis do not investigate the impact of prediction. Therefore, we still lack a thorough theoretical understanding about the value-of-prediction in stochastic network control. Fundamental questions regarding how prediction should be integrated in network algorithms, the ultimate prediction gains, and how prediction error impacts performance, remain largely unanswered.

To contribute to developing a theoretical foundation for utilizing prediction in networks, in this paper, we consider a general constrained stochastic network optimization formulation, and aim to rigorously quantify the benefits of system state prediction and the impact of prediction error. Specifically, we are given a discrete-time stochastic network with a dynamic state that evolves according to some potentially non-stationary probability law. Under each system state, a control action is chosen and implemented. The action generates traffic into network queues but also serves workload from them. The action also results in a system utility (cost) due to service completion (resource expenditure). The traffic, service, and cost are jointly determined by the action and the system state. The objective is to maximize the expected utility (or equivalently, to minimize the cost) subject to traffic/service constraints, given imperfect system state prediction information.

This is a general framework that models various practical scenarios, for instance, mobile networks, computer networks, supply chains, and smart grids. However, understanding the impact of prediction in this framework is challenging. First, statistical information of network dynamics is often unknown a-priori. Hence, in order to achieve good performance, algorithms must be able to quickly learn certain sufficient statistics of the dynamics, and make efficient use of prediction while carefully handling prediction error. Second, system states appear randomly in every time slot. Thus, algorithms must perform well under such incremental realizations of the randomness. Third, quantifying system service quality often involves handling queuing in the system. As a result, explicit connections between control actions and queues must be established.

There has been a recent effort in developing algorithms that can achieve good utility and delay performance for this general problem without prediction in various settings, for instance, wireless networks, [7]–[10], processing
networks, [11], [12], cognitive radio, [13], and the smart grid, [14], [15]. However, existing results mostly focus on networks with stationary distributions. They either assume full system statistical information beforehand, or rely on stochastic approximation techniques to avoid the need of such information. Huang et al. [16] and Huang [17] propose schemes to incorporate historic system information into control, but they do not consider prediction. Recent results in [18]–[22] consider problems with traffic demand prediction, and Muppipisetty et al. [23] jointly consider demand and channel prediction. However, they focus either on $M/M/1$-type models, or do not consider queueing, or do not consider the impact of prediction error. Along a different line of work, Zhao et al. [24], Chen et al. [25], [26], and Hajiesmaili et al. [27] investigate the benefit of prediction from the online algorithm design perspective. Although the results provide new understanding about the effect of prediction, they do not apply to the general constrained network optimization problem in consideration, where action outcomes are general functions of time-varying network states, queues evolve in a controlled manner, i.e., arrival and departure rates depend on the control policy, and prediction can contain error.

In this paper, we develop a novel control algorithm for the general framework called predictive learning-aided control (PLC). PLC is an online scheme that consists of three components, sequential distribution estimation and change detection, dual learning, and online control (see Fig. 1).

The distribution estimator conducts sequential statistical comparisons based on prediction and historic network state records. Doing so efficiently detects changes of the underlying probability distribution and guides us in selecting the right state samples to form distribution estimates. The estimated distribution is then fed into a dual learning component to compute an empirical multiplier of an underlying optimization formulation. This multiplier is further incorporated into the Backpressure (BP) controller [1] to perform realtime network operation. Compared to the commonly adopted receding-horizon-control approach (RHC), e.g., [28], PLC provides another way to utilize future state information, which focuses on using the predicted distribution for guiding action selection in the present slot and can be viewed as performing steady-state control under the predicted future distribution.

We summarize our main contributions as follows.

i. We propose a general state prediction model featured with a distribution-accuracy curve. Our model captures key factors of several existing prediction models, including window-based [22], distribution-based [29], and filter-based [26] models.

ii. We propose a general constrained network control algorithm called predictive learning-aided control (PLC). PLC is an online algorithm that is applicable to both stationary and non-stationary systems. It jointly performs distribution estimation and change detection, dual learning, and queue-based online control.

iii. We show that for stationary networks, PLC achieves an $O(\epsilon), O(\log^2(1/\epsilon))$ utility-delay tradeoff. For non-stationary networks, PLC obtains an $O(\epsilon), O(\log^2(1/\epsilon) + \min(\epsilon^2/2 - 1, e_w/\epsilon))$ utility-backlog tradeoff for distributions that last $\Theta_{\max}(e_w^{-1}, e_w^{-1})$ time, where $e_w$ is the prediction accuracy, $c \in (0, 1)$ and $a > 0$ is an $\Theta(1)$ constant (the Backpressure algorithm [1] requires an $O(e^{-2})$ length for the same utility performance with a larger backlog).

iv. We show that for both stationary and non-stationary system dynamics, PLC detects distribution change $O(w)$ slots ($w$ is prediction window size) faster with high probability and achieves a fast $O(\min(\epsilon^{-1+c/2}, e_w/\epsilon + \log^2(1/\epsilon))$ convergence time, which is faster than the $O(\epsilon^{-1+c/2} + e^{-c})$ time of the OLAC scheme [16], and the $O(1/\epsilon)$ time of Backpressure.

The rest of the paper is organized as follows. In Section II, we discuss a few motivating examples. We set up the notations in Section III, and present the problem formulation in Section IV. Background information is provided in Section V. Then, we present PLC and its analysis in Sections VI and VII. Simulation results are presented in Section VIII, followed by conclusions in Section IX. To facilitate reading, all the proofs are placed in the appendices.

## II. Motivating Examples

In this section, we present a few interesting practical scenarios that fall into our general framework.

**Matching in sharing platforms:** Consider a Uber-like company that provides ride service to customers. At every time, customer requests enter the system and available cars join to provide service. Depending on the environment condition (state), e.g., traffic condition or customer status, matching customers to drivers can result in different user satisfaction, and affect the revenue of the company (utility). The company gets access to future customer demand and car availability, and system condition information (prediction), e.g., through reservation or machine learning tools. The objective is to optimally match customers to cars so that the utility is maximized, e.g., [30] and [31].

**Energy optimization in mobile networks:** Consider a basestation (BS) sending traffic to a set of mobile users. The channel conditions (state) between users and the BS are time-varying. Thus, the BS needs different amounts of power for packet transmission (cost) at different times. Due to higher layer application requirements, the BS is required to deliver packets to users at pre-specified rates. On the other hand, the BS can predict future user locations in some short period

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1Note that when there is no prediction, i.e., $w = 0$ and $e_w = \infty$, we recover previous results of OLAC [16].
of time, from which it can estimate future channel conditions (prediction). The objective of the BS is to jointly optimize power allocation and scheduling among users, so as to minimize energy consumption, while meeting the rate requirements, e.g., [8], [13]. Other factors such as energy harvesting, e.g., [32], can also be incorporated in the formulation.

Resource allocation in cloud computing: Consider an operator, e.g., a dispatcher, assigning computing jobs to servers for processing. The job arrival process is time-varying (state), and available processing capacities at servers are also dynamic (state), e.g., due to background processing. Completing users' job requests brings the operator reward (utility). The operator may also have information regarding future job arrivals and service capacities (prediction). The goal is to allocate resources and to balance the loads properly, so as to maximize system utility. This example can be extended to capture other factors such as rate scaling [33] and data locality constraints [34].

In these examples and related works, not only can the state statistics be potentially non-stationary, but the systems also often get access to certain (possibly imperfect) future state information through various prediction techniques. These features make the problems different from existing settings considered, e.g., [8] and [15], and require different approaches for both algorithm design and analysis.

III. NOTATIONS

\( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space. \( \mathbb{R}_+^n \) denotes the non-negative (non-positive) orthant. Bold symbols \( \mathbf{x} = (x_1, \ldots, x_n) \) denote vectors in \( \mathbb{R}^n \). \( w.p.1 \) denotes “with probability 1.” \( \| \cdot \| \) denotes the Euclidean norm. For a sequence \( \{y(t)\}_{t=0}^\infty \), \( \mathbb{E} = \lim_{t \to \infty} \frac{1}{t} \sum_{t=0}^{t-1} \mathbb{E} \{y(\tau)\} \) denotes its average (when exists). \( \mathbf{x} \geq \mathbf{y} \) means \( x_j \geq y_j \) for all \( j \). For distributions \( \pi_1 \) and \( \pi_2 \), \( \| \pi_1 - \pi_2 \|_{TV} = \sum_i |\pi_{1i} - \pi_{2i}| \) denotes the total variation distance.

IV. SYSTEM MODEL

Consider a controller that operates a network with the goal of minimizing the time average cost, subject to the queue stability constraint. The network operates in slotted time, i.e., \( t \in \{0, 1, 2, \ldots\} \), and there are \( r \geq 1 \) queues in the network.

A. Network State

In every slot \( t \), \( S(t) \) denotes the current network state, which summarizes current network parameters, such as a vector of conditions for each network link, or a collection of other relevant information about the current network channels and arrivals. \( S(t) \) is independently distributed across time, and each realization is drawn from a state space of \( M \) distinct states denoted as \( S = \{s_1, s_2, \ldots, s_M\} \). We denote \( \pi_i(t) = \Pr\{S(t) = s_i\} \) the probability of being in state \( s_i \) at time \( t \) and denote \( \pi(t) = (\pi_1(t), \ldots, \pi_M(t)) \) the state distribution. The network controller can observe \( S(t) \) at the beginning of every slot \( t \), but the \( \pi_i(t) \) probabilities are unknown. We assume that each \( \pi(t) \) stays unchanged for multiple slot timeslots, and denote \( \{t_k, k = 0, 1, \ldots\} \) the starting point of the \( k \)-th constant distribution interval \( I_k \), i.e., \( \pi(t) = \pi_k \) for all \( t \in I_k \) for all \( k \in \mathbb{N} \). The length of \( I_k \) is denoted by \( d_k = t_{k+1} - t_k \).

B. State Prediction

At every time slot, the operator gets access to a prediction module, e.g., a machine learning algorithm, which provides prediction of future network states. Different from recent works, e.g., [25], [26], and [35], which assume prediction models on individual states, we assume that the prediction module outputs a sequence of predicted distributions \( \mathcal{D}_w(t) \triangleq \{\hat{\pi}(t), \hat{\pi}(t+1), \ldots, \hat{\pi}(t+w)\} \), where \( w+1 \) is the prediction window size. Moreover, the prediction quality is characterized by a distribution-accuracy curve \( \{e(0), \ldots, e(w)\} \) as follows. For every \( 0 \leq k \leq w \), \( \hat{\pi}(t + k) \) satisfies:

\[ ||\pi(t+k) - \hat{\pi}(t+k)||_{TV} \leq e(k), \forall k. \]

That is, the predicted distribution at time \( k \) has a total-variation error bounded by some \( e(k) \geq 0 \).\(^3\) Note that \( e(k) = 0 \) for all \( 0 \leq k \leq w \) corresponds to a perfect predictor, in that it predicts the exact distribution in every slot. We assume the \( \{e(0), \ldots, e(w)\} \) curve is known to the operator and denote \( e_w \triangleq \sum_{k=0}^{w} e(k) \) the average prediction error.

It is often possible to achieve prediction guarantees as in (1), e.g., by adopting a maximum likelihood estimator [36, Sec. 5.1] based on historic state data. Also note that the prediction model in (1) is general and captures key features of several existing prediction models: (i) the exact distribution prediction model in [29], where the future demand distribution is known \( (e(k) = 0 \text{ for all } k) \), (ii) the window-based prediction model, e.g., [22], where each \( \hat{\pi}(t+k) \) corresponds to the indicator for the true state, and (iii) the error-convolution prediction model in [25], [26], and [35], which captures key features of the Wiener filter and Kalman filter.

C. The Cost, Traffic, and Service

At each time \( t \), after observing \( S(t) = s_i \), the controller chooses an action \( x(t) \in X_i \). The set \( X_i \) is called the feasible action set for network state \( s_i \) and is assumed to be time-invariant and compact for all \( s_i \in S \). The cost, traffic, and service generated by the action \( x(t) = x_i \) are as follows:

(a) The chosen action has an associated cost given by the cost function \( f(t) = f(S(t), x(t)) = f(s_i, x_i) \). \( X_i \mapsto \mathbb{R}_+ \) (or \( X_i \mapsto \mathbb{R}_- \) in reward maximization problems).\(^4\)

(b) The amount of traffic generated by the action to queue \( j \) is determined by the traffic function \( A_j(t) = A_j(S(t), x(t)) = A_j(s_i, x_i) \). \( X_i \mapsto \mathbb{R}_+ \), in units of packets.

(c) The amount of service allocated to queue \( j \) is given by the rate function \( \mu_j(t) = \mu_j(S(t), x(t)) = \mu_j(s_i, x_i) \). \( X_i \mapsto \mathbb{R}_+ \), in units of packets.

\(^3\) We focus on state distribution prediction instead of predicting individual states. In this case, it makes sense to assume a deterministic upper bound of the difference because we are dealing with distributions.

\(^4\) We use cost and utility interchangeably in this paper.
Here $A_j(t)$ can include both exogenous arrivals from outside the network to queue $j$, and endogenous arrivals from other queues, i.e., transmitted packets from other queues to queue $j$. We assume the functions $-f(s_i, \cdot)$, $\mu_j(s_i, \cdot)$ and $A_j(s_i, \cdot)$ are time-invariant, their magnitudes are uniformly upper bounded by some constant $\delta_{\text{max}} \in (0, \infty)$ for all $s_i$, $j$, and they are known to the operator. Note that this formulation is general and models many network problems, e.g., [8], [15], and [37].

**D. Problem Formulation**

Let $q(t) = (q_1(t), \ldots, q_r(t))^T \in \mathbb{R}^r_+$, $t = 0, 1, 2, \ldots$ be the queue backlog vector process of the network, in units of packets. We assume the following queueing dynamics:

$$q_j(t + 1) = \max \left[ q_j(t) - \mu_j(t) + A_j(t), 0 \right], \quad \forall j,$$

and $q(0) = 0$. By using (2), we assume that when a queue does not have enough packets to send, null packets are transmitted, so that the number of packets entering $q_j(t)$ is equal to $A_j(t)$.

We adopt the following notion of queue stability [1]:

$$\bar{q}_{\text{av}} \triangleq \lim_{t \to \infty} \sup_t \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{j=1}^{r} \mathbb{E} \{ q_j(\tau) \} < \infty. \quad (3)$$

We use $\Pi$ to denote an action-choosing policy, and use $f_{\text{av}}^\Pi$ to denote its time average cost, i.e.,

$$f_{\text{av}}^\Pi \triangleq \lim_{t \to \infty} \sup_t \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{j=1}^{r} \mathbb{E} \{ f^\Pi(\tau) \}, \quad (4)$$

where $f^\Pi(\tau)$ is the cost incurred at time $\tau$ under policy $\Pi$. We call an action-choosing policy feasible if at every time slot $t$ it only chooses actions from the feasible action set $\mathcal{X}_i$ when $S(t) = s_i$. We then call a feasible action-choosing policy under which (3) holds a stable policy.

In every slot, the network controller observes the current network state and prediction, and chooses a control action, with the goal of minimizing the time average cost subject to network stability. This goal can be mathematically stated as:

\[
\text{(P1)} \quad \min_{\Pi} f_{\text{av}}^\Pi, \quad \text{s.t.} \quad (3).
\]

In the following, we call (P1) the stochastic problem, and we use $f_{\text{av}}^\Pi$ to denote its optimal solution given a fixed distribution $\pi$. It can be seen that the examples in Section II can all be modeled by our stochastic problem framework.

Throughout our paper, we make the following assumption:

**Assumption 1:** For every system distribution $\pi_k$, there exists a constant $\epsilon_k = \Theta(1) > 0$ such that for any valid state distribution $\pi' = (\pi'_1, \ldots, \pi'_M)$ with $\| \pi' - \pi_k \|_1 \leq \epsilon_k$, there exist a set of actions $\{ x_{(s_i)}^{(s_i, z)} : 1, 2, \ldots, M \}$ with $x_{(s_i)}^{(s_i, z)} \in \mathcal{X}_i$ and variables $\alpha_{(s_i)}^{(s_i, z)} \geq 0$ for all $s_i$ and $z$ with $\sum_{s_i} \alpha_{(s_i)}^{(s_i, z)} = 1$ for all $s_i$ (possibly depending on $\pi'$), such that:

$$\sum_{s_i} \pi'_i \left\{ \sum_{z} g_{(s_i)}^{(s_i, z)} \left[ A_j(s_i, x_{(s_i)}^{(s_i, z)}) - \mu_j(s_i, x_{(s_i)}^{(s_i, z)}) \right] \right\} \leq -\eta_0, \quad \forall j, \quad (5)$$

where $\eta_0 = \Theta(1) > 0$ is independent of $\pi'$. \quad \diamond

5When $\pi(t)$ is time-varying, the optimal system utility needs to be defined carefully. We will specify it when discussing the corresponding results.

Assumption 1 corresponds to the “slack” condition commonly assumed in the literature with $\epsilon_k = 0$, e.g., [37] and [38].6 With $\epsilon_k > 0$, we assume that when two systems are relatively close to each other (in terms of $\pi$), they can both be stabilized by some (possibly different) randomized control policy that results in the same slack.

**E. Discussion of the Model**

Two key differences between our model and previous ones include (i) $\pi(t)$ itself can be time-varying and (ii) the operator gets access to a prediction window $\mathcal{W}_t(w)(t)$ that contains imperfect prediction. These two extensions are important to the current network control literature. First, practical systems are often non-stationary. Thus, system dynamics can have time-varying distributions. Thus, it is important to have efficient algorithms to automatically adapt to the changing environment. Second, prediction has recently been made increasingly accurate in various contexts, e.g., user mobility in cellular network and harvestable energy availability in wireless systems, by data collection and machine learning tools. Thus, it is critical to understand the fundamental benefits and limits of prediction, and its optimal usage. Finally, note that the convexity of the problem (P1) depends largely on the structure of the feasible action sets. In the case, when all feasible action sets are convex, it can be shown that the resulting problem is convex using a similar argument as that in [39].

**V. THE DETERMINISTIC PROBLEM**

For our later algorithm design and analysis, we define the deterministic problem and its dual problem [40]. Specifically, the deterministic problem for a given distribution $\pi$ is defined as follows [40]:

\[
\begin{align*}
\text{min} & : V \sum_{s_i} \pi_i f(s_i, x_{(s_i)}) \\
\text{s.t.} & \sum_{s_i} \pi_i \left[ A_j(s_i, x_{(s_i)}) - \mu_j(s_i, x_{(s_i)}) \right] \leq 0, \quad \forall j, \\
x_{(s_i)} \in \mathcal{X}_i & \quad \forall i = 1, 2, \ldots, M.
\end{align*}
\]

Here the minimization is taken over $x \in \prod_{i} \mathcal{X}_i$, where $x = (x_{(s_i)}, \ldots, x_{(s_M)})^T$, and $V \geq 1$ is a positive constant introduced for later analysis. The dual problem of (6) can be obtained as follows:

\[
\begin{align*}
\max & : g(\gamma, \pi), \quad \text{s.t.} \quad \gamma \succeq 0, \\
\text{where } g(\gamma, \pi) & = \inf_{x_{(s_i)} \in \mathcal{X}_i} \sum_{s_i} \pi_i \left\{ V f(s_i, x_{(s_i)}) \\
& \quad + \sum_j \gamma_j \left[ A_j(s_i, x_{(s_i)}) - \mu_j(s_i, x_{(s_i)}) \right] \right\},
\end{align*}
\]

\[
\gamma = (\gamma_1, \ldots, \gamma_r)^T \text{ is the Lagrange multiplier of (6). It is well known that } g(\gamma, \pi) \text{ in (8) is concave in the vector } \gamma \text{ for all } \gamma \in \mathbb{R}^r. \text{ Hence, the problem (7) can usually be solved}
\]

6Note that $\eta_0 \geq 0$ is a necessary condition for network stability [1].
efficiently, e.g., using dual subgradient methods [41]. If the cost functions and rate functions are separable over different network components, the problem also admits distributed solutions [41]. We use $\gamma_{\pi}$ to denote the optimal multiplier corresponding to a given $\pi$ and sometimes omit the subscript when it is clear. Denote $g^*_\pi$ the optimal value of (7) under a fixed distribution $\pi$. It was shown in [42] that:

$$
    f^*_{av} = g^*_{\pi}.
$$

That is, $g^*_{\pi}$ characterizes the optimal time average cost of the stochastic process. For our analysis, we make the following assumption on the $g(\gamma, \pi_k)$ function.

Assumption 2: For every system distribution $\pi_k$, $g(\gamma, \pi_k)$ has a unique optimal solution $\gamma_{\pi_k} \neq 0$ in $\mathbb{R}^7$.

VI. PREDICTIVE LEARNING-AIDED CONTROL

In this section, we present the predictive learning-aided control algorithm (PLC). PLC contains three main components: a distribution estimator, a learning component, and an online queue-based controller. Below, we first present the estimation part. Then, we present the PLC algorithm.

A. Distribution Estimation and Change Detection

Here we specify the distribution estimator. The idea is to first combine the prediction in $\mathcal{W}_d(t)$ with historic state information to form an average distribution, and then perform statistical comparisons for change detection. We call this module the average distribution estimate (ADE).

Specifically, ADE maintains two windows $\mathcal{W}_m(t)$ and $\mathcal{W}_d(t)$ to store network state samples, where $\mathcal{W}_d(t)$ roughly contains the most recent $d$ state samples, and $\mathcal{W}_m(t)$ contains at most $T_1$ state samples after $\mathcal{W}_d(t)$. The formal definition of them are given below, i.e.,

$$
    \mathcal{W}_d(t) = \{b_d^0(t), \ldots, b_d^d(t)\},
$$

$$
    \mathcal{W}_m(t) = \{b_m(t), \ldots, \min[b_d^0(t), b_d^m(t) + T_1]\}.
$$

Here $b_d^0(t)$ and $b_m(t)$ mark the beginning slots of $\mathcal{W}_d(t)$ and $\mathcal{W}_m(t)$, respectively, and $b_d^d(t)$ marks the end of $\mathcal{W}_d(t)$. Ideally, $\mathcal{W}_d(t)$ contains the most recent $d$ samples (including the prediction) and $\mathcal{W}_m(t)$ contains $T_1$ subsequent samples (where $T_1$ is a pre-specified number). We denote $W_m(t) = |\mathcal{W}_m(t)|$ and $W_d(t) = |\mathcal{W}_d(t)|$. Without loss of generality, we assume that $d \ge w + 1$. This assumption is made because, $d$ grows with our control parameter $V$ while prediction power is often limited in practice. We also denote $W_d(t) \equiv \{t, \ldots, t + w\}$.

We use $\bar{\pi}^d(t)$ and $\bar{\pi}^m(t)$ to denote the empirical distributions of $\mathcal{W}_d(t)$ and $\mathcal{W}_m(t)$, i.e.,

$$
    \bar{\pi}^d(t) = \frac{1}{d} \sum_{\tau=(t+w-d)}^{t-1} 1_{S(\tau) = s_i} + \sum_{\tau \in \mathcal{W}_d(t)} \bar{\pi}_i(\tau),
$$

$$
    \bar{\pi}^m(t) = \frac{1}{W_m(t)} \sum_{\tau \in \mathcal{W}_m(t)} 1_{S(\tau) = s_i}.
$$

Note that this is only one way to utilize the samples. Other methods such as EWMA can also be applied when appropriate.

That is, $\bar{\pi}^d(t)$ is the average of the empirical distribution of the “observed” samples in $\mathcal{W}_d(t)$ and the predicted distribution, whereas $\bar{\pi}^m(t)$ is the empirical distribution of $\mathcal{W}_m(t)$.

The formal procedure of ADE is as follows (parameters $T_1, d, \epsilon_d$ will be specified later).

Average Distribution Estimate (ADE($T_1, d, \epsilon_d$)): Initialize $b_d^0(0) = 0$, $b_d^d(0) = t + w$ and $b_m(0) = 0$, i.e., $\mathcal{W}_d(t) = \{0, \ldots, t + w\}$ and $\mathcal{W}_m(t) = \phi$. At every time $t$, update $b_d^0(t)$, $b_d^d(t)$ and $b_m(t)$ as follows:

(i) If $W_m(t) \ge d$ and $||\bar{\pi}^d(t) - \bar{\pi}^m(t)||_V > \epsilon_d$, set $b_m(t) = t + w + 1$ and $b_d^d(t) = b_d^d(t) + w + 1$.

(ii) If $W_m(t) = T_1$ and there exists $k$ such that $||\bar{\pi}(t + k) - \bar{\pi}^m(t)||_V > \epsilon(k) + 2M \log(T_1)$, set $b_m(t) = b_d^d(t) = t + w + 1$. Mark $t + w + 1$ a reset point.

(iii) Else if $t \le b_d^d(t - 1)$, $b_m(t) = b_m(t - 1)$, $b_d^d(t) = b_d^d(t - 1) - 1$, $b_d^d(t) = b_d^d(t) + 1$.

(iv) Else set $b_m(t) = b_m(t - 1)$, $b_d^d(t) = (t + w - d) + 1$, $b_d^d(t) = t + w$.

Output an estimate at time $t$ as follow:

$$
    \pi_\alpha(t) = \begin{cases} 
    \bar{\pi}^m(t) & \text{if } W_m(t) \ge T_1 \\
    \frac{W_m(t)}{w+1} \sum_{k=0}^{w} \bar{\pi}(t + k) & \text{else}
    \end{cases}
$$

The idea of ADE is shown in Fig. 2.

The intuition of ADE is that if the environment is changing over time, we should rely on prediction for control. Else if the environment is stationary, then one should use the average distribution learned over time to combat the potential prediction error that may affect performance. $T_1$ is introduced to ensure the accuracy of the empirical distribution and can be regarded as the confidence-level given to the distribution stationarity. A couple of technical remarks are also ready.

(a) The term $2M \log(T_1)/\sqrt{T_1}$ is to compensate the inevitable deviation of $\bar{\pi}^m(t)$ from the true value due to randomness.

(b) In $\mathcal{W}_m(t)$, we only use the first $T_1$ historic samples. Doing so avoids random oscillation in estimation and facilitates analysis.

Prediction is used in two ways in ADE. First, it is used in step (i) to decide whether the empirical distributions match (average prediction). Second, it is used to check whether prediction is consistent with the history (individual prediction). The reason for having this two-way utilization is to accommodate general prediction types. For example, suppose each $\pi(t + k)$ denotes the indicator for state $S(t + k)$, e.g., as in the

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{fig2}
  \caption{Evolution of $\mathcal{W}_d(t)$ and $\mathcal{W}_d(t)$. (Left) No change detected: The samples satisfy the criteria in Steps (i) and (ii). Thus, $\mathcal{W}_d(t)$ advances by one slot and $\mathcal{W}_d(t)$ increases its size by one. (Right) Change detected: The samples violate at least one of the conditions in (i) and (ii). Thus, both windows set their start and end points to $t + w + 1$.}
\end{figure}
look-ahead window model [22]. Then, step (ii) is loose since $e(k)$ is large, but step (i) will be useful. If $\bar{\pi}(t+k)$ gets closer to the true distribution, both steps will be useful.

B. Predictive Learning-Aided Control

We are now ready to present the PLC algorithm (shown in Fig. 1). The formal description is given below.

**Predictive Learning-aided Control (PLC):** At time $t$, do:

1) (Estimation) Update $\pi_a(t)$ with $\text{ADE}(T_1, d, \epsilon_d)$.

2) (Learning) Solve the following empirical problem and compute the optimal Lagrange multiplier $\gamma^*(t)$, i.e.,

$$\max : \quad g(\gamma, \pi_a(t)), \quad \text{s.t.} \quad \gamma \geq 0,$$

If $\gamma^*(t) = \infty$, set $\gamma^*(t) = V \log(V) \cdot 1$. If $W_{m}(t-1) = T_1$ and $\pi_a(t) \neq \pi_a(t-1)$, set $q(t+w+1) = 0$, i.e., drop all packets currently in the queues.

3) (Control) At every time slot $t$, observe the current network state $S(t)$ and the backlog $q(t)$. If $S(t) = s_i$, choose $x(s_i) \in X_t$ that solves the following:

$$\max : \quad -V f(s_i, x) + \sum_{j=1}^{r} Q_j(t) \left[ \mu_j(s_i, x) - A_j(s_i, x) \right],$$

s.t. $x \in X_t$.

where $Q_j(t) \triangleq q_j(t) + (\gamma^*_j(t) - \theta)^+$.

Then, update the queues according to (2) (Last-In-First-Out).

For readers who are familiar with the Backpressure (BP) algorithm, e.g., [1] and [44], the control component of PLC is the BP algorithm with its queue vector augmented by the empirical multiplier $\gamma^*(t)$. Also note that packet dropping is introduced to enable quick adaptation to new dynamics if there is a distribution change. It occurs only when a long-lasting distribution ends, which avoids dropping packets frequently in a fast-changing environment.

We have the following remarks. (i) **Prediction usage:** Prediction is explicitly incorporated into control by forming an average distribution and converting the distribution estimate into a Lagrange multiplier. The intuition for having $T_1 = \max(V^c, e_w^{-2})$ is that when $e_w$ is small, we should rely on prediction as much as possible, and only switch to learning statistics when it is sufficiently accurate. (ii) **Connection with RHC:** It is interesting to see that when $W_{m}(t) < T_1$, PLC mimics the commonly adopted receding-horizon-control method (RHC), e.g., [28]. The main difference is that, in RHC, future states are predicted and are directly fed into a predictive optimization formulation for computing the current action. Under PLC, *distribution prediction* is combined with *historic state information* to compute an empirical multiplier for augmenting the controller. In this regard, PLC can be viewed as exploring the benefits of statistics whenever it finds the system stationary (and does it automatically). (iii) **Parameter selection:** The parameters in PLC can be conveniently chosen as follows. First, fix a detection error probability $\delta = V^{-\log(V)}$. Then, choose a small $\epsilon_d$ and a $d$ that satisfies $d \geq 4 \log(V)^2/e_d^2 + w + 1$. Finally, choose $T_1 = \max(V, e_w^{-2})$ and $\theta$ according to (16).

While recent works [16] and [17] also design learning-based algorithms that utilize historic information, they do not consider prediction and do not provide insight on its benefits and the impact of prediction error. Moreover, Huang et al. [16] focus on stationary systems and Huang [17] adopts a frame-based scheme.

VII. PERFORMANCE ANALYSIS

This section presents the performance results of PLC. We focus on four metrics, detection efficiency, network utility, service delay, and algorithm convergence. The metrics are chosen to represent robustness, resource utilization efficiency, quality-of-service, and adaptability, respectively. Since our main objective is to investigate how these metrics behave under our algorithm, we treat network parameters $M$ and $r$ as constants.

A. Detection and Estimation

We first look at the detection and estimation part. The following lemma summarizes the performance of ADE, which is affected by the prediction accuracy as expected.

**Lemma 1:** Under $\text{ADE}(T_1, d, \epsilon_d)$, we have:

(a) Suppose at a time $t$, $\pi(\tau_1) = \pi_1$ for $\tau_1 \in W_d(t)$ and $\pi(\tau_2) = \pi_2$ for $\tau_2 \in W_m(t)$ and $\max(|\pi_1 - \pi_2| > 4(w+1)e_w/d$. Then, by choosing $\epsilon_0 > \delta$ and $d > \max_{\pi_1, \pi_2} [\pi_1 \cdot \pi_2] > 2 + (w+1)e_w/d$ with probability at least $1 - \delta$, $b_{m}(t+1) = t+w+1$ and $W_{m}(t+1) = 0$, i.e., $W_{m}(t+1) = 0$.

(b) Suppose $\pi(t) = \pi \forall t$. Then, if $W_{m}(t) \geq W_d(t) = d$, under $\text{ADE}(T_1, d, \epsilon_d)$ with $d \geq \max_{\pi_1, \pi_2} [\pi_1 \cdot \pi_2] > 4(w+1)e_w/d$ can be understood as follows. If we want to distinguish two different distributions, we want the detection threshold to be no more than half of the distribution distance. Now with prediction, we want the potential detection error to be no more than half of the threshold, hence the factor 4. Also note that the delay involved in detecting a distribution change is nearly order-optimal, i.e., $d = O(1/\min(|\pi_1 - \pi_2|^2)$ time, which is known to be necessary for distinguishing two distributions [45]. Moreover, $d = O(\log(1/\delta))$ shows that a logarithmic window size is enough to ensure a high detection accuracy.

B. Utility and Delay

In this section, we look at the utility and delay performance of PLC. To state our results, we first define the following structural property of the system.
Definition 1: A system is called polyhedral with parameter \( \rho > 0 \) under distribution \( \pi \) if the dual function \( g(\gamma, \pi) \) satisfies:
\[
g(\gamma^*, \pi) \geq g(\gamma, \pi) + \rho \| \gamma^* - \gamma \|. 
\]
(15)

The polyhedral property often holds for practical systems, especially when action sets are finite (see [40] for more discussions).

1) Stationary System: We first consider stationary systems, i.e., \( \pi(t) = \pi. \) Our theorem shows that PLC achieves the near-optimal utility-delay tradeoff for stationary networks. This result is important, as any good adaptive algorithm must be able to handle stationary settings well.

Theorem 1: Suppose \( \pi(t) = \pi \), the system is polyhedral with \( \rho = \Theta(1) \), \( e_w > 0 \), and \( q(0) = 0. \) Choose \( 0 < \epsilon_d < \epsilon_0 \equiv 2(w + 1) / e_w / d, d = \log(V)^3 / e_w^2, T = \max(V^c, e_w^{-2}) \) for \( c \in (0, 1) \) and
\[
\theta = 2 \log(V)^2 (1 + \frac{V}{\sqrt{T}}). 
\]
(16)

Then, with a sufficiently large \( V \), PLC achieves the following:
(a) Utility: \( f^{\text{PLC}}_w = f^{\pi_\text{av}} + O(1 / V) \)
(b) Delay: For all but an \( O(1 / V) \) fraction of the traffic, the average packet delay is \( D = O(\log(V)^2) \)
(c) Dropping: The packet dropping rate is \( O(V^{-1}) \).

Proof: Omitted due to space limitation. Please see [46] for proof details.

Here \( c \) is a constant used for deciding the learning time \( T \). Choosing \( c = 1 / V \), we see that PLC achieves the near-optimal \( O(\epsilon), O(\log(1/\epsilon^2)) \) utility-delay tradeoff. Moreover, prediction enables PLC to also greatly reduce the queue size (see Part (b) of Theorem 2). Our result is different from the results in [20] and [22] for proactive service settings, where delay vanishes as prediction power increases. This is because we only assume observability of future states but not pre-service, and highlights the difference between pre-service and pure prediction. Note that the performance of PLC does not depend heavily on \( \epsilon_d \) in Theorem 1. The value \( \epsilon_d \) is more crucial for non-stationary systems, where a low false-negative rate is critical for performance. Also note that although packet dropping can occur during operation, the fraction of packets dropped is very small, and the resulting performance guarantee cannot be obtained by simply dropping the same amount of packets, in which case the delay will still be \( \Theta(1/\epsilon) \).

Although Theorem 1 has a similar form as those in [16] and [17], the analysis is very different, in that (i) prediction error must be taken into account, and (ii) PLC performs sequential and decision-making.

2) Piecewise Stationary System: We now turn to the non-stationary case and consider the scenario where \( \pi(t) \) changes over time. In this case, we see that prediction is critical as it significantly accelerates convergence and helps to achieve good performance when each distribution only lasts for a finite time. As we know that when the distribution can change arbitrarily, it is hard to even define optimality. Thus, we consider the case when the system is piecewise stationary, i.e., each distribution lasts for a duration of time, and study how the algorithm optimizes the performance for each distribution.

The following theorem summarizes the performance of PLC in this case. In the theorem, we define \( D_k = t_k + d - t^* \), where \( t^* = \sup \{ t < t_k + d : t \) is a reset point \}, i.e., the most recent time when a cycle with size no smaller than \( T_I \) ends (recall that reset points are marked in step (ii) of ADE and \( d \geq w + 1 \)).

Theorem 2: Suppose \( d_k \geq 4d \) and the system is polyhedral with \( \rho = \Theta(1) \) for all \( k \). Also, suppose there exists \( \epsilon_0 = \Theta(1) > 0 \) such that \( \epsilon_0 \leq \inf_{k,i} |\pi_{k,i} - \pi_{k-1,i}| \) and \( q(0) = 0. \) Choose \( \epsilon_d < \epsilon_0 \) in ADE, and choose \( d, \theta \) and \( T_I \) as in Theorem 1. Fix any distribution \( \pi_k \) with length \( d_k = \Theta(V^{1+a}T_I) \) for some \( a = \Theta(1) > 0 \). Then, under PLC with a sufficiently large \( V \), if \( W_m(t_k) \) only contains samples after \( t_{k-1} \), we achieve the following with probability \( 1 - O(V^{-3 \log(V)/4}) \):
(a) Utility: \( f^{\text{PLC}}_w = f^{\pi_\text{av}} + O(1 / V) + O(D_w / V^{1+a}) \)
(b) Queueing: \( \bar{T}_{w} = O((\min(V^{1-c/2}, V e_w) + 1) \log^2(V) + D_k + d) \)
(c) In particular, if \( d_{k-1} = \Theta(T_I V^{a_k}) \) for \( a_1 = \Theta(1) > 0 \) and \( W_m(t_k) \) only contains samples after \( t_{k-2} \), then with probability \( 1 - O(V^{-2}) \), \( D_w = O(d), f^{\text{PLC}}_w = f^{\pi_\text{av}} + O(1 / V) \) and \( \bar{T}_{w} = O((\min(V^{-c/2}, V e_w) + 1) \log^2(V) \).

Proof: Omitted due to space limitation. Please see [46] for proof details.

A few remarks are in place. (i) Theorem 2 shows that, with an increasing prediction power, i.e., a smaller \( e_w \), it is possible to simultaneously reduce network queue size and the time it takes to achieve a desired average performance (even if we do not execute actions ahead of time). The requirement \( d_k = \Theta(V^{1+a}T_I) \) can be strictly less than the \( O(V^{2-c/2+a}) \) requirement for PLC in [17] and the \( O(V^2) \) requirement of BP for achieving the same average utility. This implies that PLC finds a good system operating point faster, a desirable feature for network algorithms. (ii) The dependency on \( D_k \) is necessary. This is because PLC does not perform packet dropping if previous intervals do not exceed length \( T_I \). As a result, the accumulated backlog can affect decision making in the current interval. Fortunately the queues are shown to be small and do not heavily affect performance (also see simulations). (iii) To appreciate the queueing result, note that BP (without learning) under the same setting will result in an \( O(V) \) queue size.

Compared to the analysis in [17], one complicating factor in proving Theorem 2 is that ADE may not always throw away samples from a previous interval. Instead, ADE ensures that with high probability, only \( o(d) \) samples from a previous interval will remain. This ensures high learning accuracy and fast convergence of PLC. One interesting special case not covered in the last two theorems is when \( e_w = 0. \) In this case, prediction is perfect and \( T_I = \infty \), and PLC always runs with \( \pi_a(t) = \frac{1}{w} \sum_{k=0}^{w} \pi(t + k) \), which is the exact average distribution. For this case, we have the following result.

Theorem 3: Suppose \( e_w = 0 \) and \( q(0) = 0. \) Then, PLC achieves the following:

10The constant \( a \) here is introduced to show that our results hold as long as \( d_k \) is larger than \( O(V T_I) \).
(a) Suppose \( \pi(t) = \pi \) and the system is polyhedral with \( \rho = \Theta(1) \). Then, under the conditions of Theorem 1, PLC achieves the \([O(\epsilon), O(\log(1/\epsilon)^2)]\) utility-delay tradeoff.

(b) Suppose \( d_k \geq d \log^2(V) \) and the system is polyhedral with \( \rho = \Theta(1) \) under each \( \pi_k \). Under the conditions of Theorem 2, for an interval \( d_k \geq V^{1+\epsilon} \) for any \( \epsilon > 0 \), PLC achieves that \( f^\text{PLC} = f^\pi + O(1/V) \) and \( \mathbb{E}\{q(t_k)\} = O(\log^4(V)). \)

Proof: Omitted due to space limitation. Please see [46] for proof details.

The intuition is that since prediction is perfect, i.e., \( \pi_w(t) = \pi_k \) during \( [t_k + d, t_{k+1} - w] \). Therefore, a better performance can be achieved. The key challenge in this case is that PLC does not perform any packet dropping. Thus, queues can build up and one needs to show that the queues will be concentrating around \( \theta \cdot 1 \) even when the distribution changes.

### C. Convergence Time

We now consider the algorithm convergence time, which is an important evaluation metric and measures how long it takes for an algorithm to reach its steady-state. While recent works [16], [17], [47], and [48] also investigate algorithm convergence time, they do not consider utilizing prediction in learning and do not study the impact of prediction error.

To formally state our results, we adopt the following definition of convergence time from [16].

**Definition 2**: Let \( \zeta > 0 \) be a given constant and let \( \pi \) be a system distribution. The \( \zeta \)-convergence time of a control algorithm, denoted by \( T_\zeta \), is the time it takes for the effective queue vector \( Q(t) \) to get to within \( \zeta \) distance of \( \gamma^* \), i.e.,

\[
T_\zeta \triangleq \inf\{t | ||Q(t) - \gamma^*|| \leq \zeta\}. \quad (17)
\]

With this definition, we have the following theorem. Recall that \( w \leq d = \Theta(\log(V)^2) \).

**Theorem 4**: Assumining all conditions in Theorem 2, except that \( \pi(t) = \pi_k \) for all \( t \geq t_k \). If \( e_w = 0 \), under PLC,

\[
\mathbb{E}\{T_G\} = O(\log^4(V)). \quad (18)
\]

Else suppose \( e_w > 0 \). Under the conditions of Theorem 2, with probability \( 1 - O(\frac{1}{V^3} + \frac{D_k}{V^3}) \),

\[
\mathbb{E}\{T_G\} = O(\theta + T_k + D_k + w) \quad (19)
\]

\[
\mathbb{E}\{T_G\} = O(d). \quad (20)
\]

Here \( G = \Theta(1) \) and \( G_1 = \Theta(D_k + 2\log(V)^2(1 + V e_w)) \), where \( D_k \) is defined in Theorem 2 as the most recent reset point before \( t_k \). In particular, if \( d_{k-1} = \Theta(T_k^a) \) for some \( a_1 = \Theta(1) > 0 \) and \( \theta = O(\log(V)^2) \), then with probability \( 1 - O(V^{-2}) \), \( D_k = O(d) \), and \( \mathbb{E}\{T_{G_1}\} = O(\log^2(V)) \).

Proof: Omitted due to space limitation. Please see [46] for proof details.

The assumption \( \pi(t) = \pi_k \) for all \( t \geq t_k \) is made to avoid the need for specifying the length of the intervals. It is interesting to compare (18), (19) and (20) with the convergence results in [16] and [17] without prediction, where it was shown

\[ T_\zeta \text{ is essentially the hitting time of the process } Q(t) \text{ to the area } \{||Q(t) - \gamma^*|| \leq \zeta\} \text{ [49].} \]

that the convergence time is \( O(V^{1-c/2} \log^2(V) + V^c) \), with a minimum of \( O(V^2/3) \). Here although we may still need \( O(V^2/3) \) time for getting into an \( G \)-neighborhood (depending on \( e_w \)), getting to the \( G_1 \)-neighborhood can take only an \( O(\log^2(V)) \) time, which is much faster compared to previous results, e.g., when \( e_w = o(V^{-2}) \) and \( D_k = O(w) \), we have \( G_1 = O(\log^2(V)) \). This confirms our intuition that prediction accelerates algorithm convergence and demonstrates the power of (even imperfect) prediction.

### VIII. Simulation

In this section, we present simulation results of PLC in a two-queue system shown in Fig. 3. Though being simple, the system models various settings, e.g., a two-user downlink transmission problem in a mobile network, a CPU scheduling problem with two applications, or an inventory control system where two types of orders are being processed.

**A1(t)**

**A2(t)**

**C1(t)**

**C2(t)**

![Fig. 3. A single-server two-queue system. Each queue receives random arrivals. The server can only serve one queue at a time.](image)

We first examine the long-term performance. Fig. 4 shows the utility-delay performance of PLC compared to BP in the stationary setting. There are two PLC we simulated, one is with \( e_w = 0 \) (PLC) and the other is with \( e_w = 0.04 \) (PLC-e). From the plot, we see that both PLCs achieve a similar utility...
We also see that during the 5000 comparison, we see that prediction error. Both versions converge much faster compared to BP.

Our results demonstrate that state prediction can help improve performance and quantify the benefits of prediction in four important metrics, i.e., utility (efficiency), delay (quality-of-service), detection (robustness), and convergence (adaptability). They provide new insight for joint prediction, learning and optimization in stochastic networks.

**APPENDIX A - PROOF OF LEMMA 1**

(Proof of Lemma 1) We prove the performance of ADE($T_1$, $d$, $\epsilon$) with an argument inspired by Bifet and Gavaldá [50]. We make use of the following concentration result.

**Theorem 5:** [31] Let $X_1$, ..., $X_M$ be independent random variables with $Pr\{X_i=1\} = p_i$ and $Pr\{X_i=0\} = 1 - p_i$. Consider $X = \sum_{i=1}^{M} X_i$ with expectation $E\{X\} = \sum_{i=1}^{M} p_i$. Then, we have:

$$Pr\{X \leq E\{X\} - m\} \leq e^{-\frac{m^2}{2d}}.$$

$$Pr\{X \geq E\{X\} + m\} \leq e^{-\frac{m^2}{2d}}.$$

(Proof) (Lemma 1) (Part (a)) In this case, it suffices to check condition (i) in ADE. Define

$$\hat{\pi}_i^d(t) \triangleq \frac{1}{d} \left( \sum_{\tau=(t+w-d)_+}^{t} 1_{[S(\tau)=s_i]} + \sum_{\tau \in \mathcal{W}_d(t)} \pi_\tau(t) \right),$$

i.e., $\hat{\pi}_i^d(t)$ is defined with the true distributions in $\mathcal{W}_d(t)$. Denote $\epsilon_1 = (w+1)e_d/d$, we see then $||\hat{\pi}_i^d(t) - \hat{\pi}_i^d(t)|| \leq \epsilon_1$. Thus, for any $\epsilon > 0$, we have:

$$Pr\{||\hat{\pi}_i^d(t) - \hat{\pi}_i^m(t)||_\mathcal{V} \leq \epsilon\} \leq Pr\{||\hat{\pi}_i^d(t) - \hat{\pi}_i^m(t)||_\mathcal{V} \leq \epsilon + \epsilon_1\} \leq Pr\{||\hat{\pi}_i^d(t) - \hat{\pi}_i^m(t)||_\mathcal{V} \leq \epsilon + \epsilon_1\}.$$ (24)

Choose $\epsilon = \frac{\epsilon_1}{2}$ and let $\epsilon_0 = \epsilon + \epsilon_1$. Fix $\alpha \in (0, 1)$ and consider $i \in \arg\max_{i}\{|\pi_{1i} - \pi_{2i}|\}$. We have:

$$Pr\{||\hat{\pi}_i^d(t) - \hat{\pi}_i^m(t)||_\mathcal{V} \leq \epsilon_0\} \leq Pr\{||\hat{\pi}_i^d(t) - \hat{\pi}_{1i}\|_\mathcal{V} \geq \alpha\epsilon_0\} \cup \{||\hat{\pi}_i^m(t) - \hat{\pi}_{2i}\|_\mathcal{V} \geq (1-\alpha)\epsilon_0\} \leq Pr\{||\hat{\pi}_i^d(t) - \hat{\pi}_{1i}\|_\mathcal{V} \geq \alpha\epsilon_0\} + Pr\{||\hat{\pi}_i^m(t) - \hat{\pi}_{2i}\|_\mathcal{V} \geq (1-\alpha)\epsilon_0\}.$$ (25)

Here the first inequality follows because if we have both $||\hat{\pi}_i^d(t) - \hat{\pi}_{1i}\|_\mathcal{V} < \alpha\epsilon_0$ and $||\hat{\pi}_i^m(t) - \hat{\pi}_{2i}\|_\mathcal{V} < (1-\alpha)\epsilon_0$, and $||\hat{\pi}_i^d(t) - \hat{\pi}_i^m(t)||_\mathcal{V} \leq \epsilon_0$, we must have:

$$|\pi_{1i} - \pi_{2i}| \leq |\hat{\pi}_i^d(t) - \hat{\pi}_{1i}| + |\hat{\pi}_i^m(t) - \hat{\pi}_{2i}| + |\hat{\pi}_i^d(t) - \hat{\pi}_i^m(t)|$$

$$= 2\epsilon_0 < |\pi_{1i} - \pi_{2i}|.$$
which contradicts the fact that $i$ achieves $\max_i |\pi_{1i} - \pi_{2i}|$. Using (25) and Hoeffding inequality [52], we first have:

$$\Pr \{ |\hat{\pi}_m^i(t) - \pi_{2i}| \geq (1 - \alpha)\epsilon_0 \} \leq 2 \exp(-2((1 - \alpha)\epsilon_0)^2 W_m(t)). \quad (26)$$

For the first term in (25), we have:

$$\Pr \{ |\hat{\pi}_i^d(t) - \pi_{1i}| \geq \alpha \epsilon_0 \} \leq 2 \exp(-2(\alpha \epsilon_0)^2 (W_d(t) - w - 1)). \quad (27)$$

Equating the above two probabilities and setting the sum equal to $\delta$, we have $\alpha = \frac{\sqrt{W_m(t)/(W_d(t)-w-1)}}{1+\sqrt{W_m(t)/(W_d(t)-w-1)}}$, and

$$\epsilon_0 = \sqrt{\frac{4}{\delta}} \frac{1 + \sqrt{(W_d(t) - w - 1)/W_m(t)}}{2(W_d(t) - w - 1)}. \quad (28)$$

In order to detect the different distributions, we can choose $\epsilon_d < \epsilon_0$, which on the other hand requires that:

$$\epsilon_d \leq \sqrt{\frac{4}{\delta}} \frac{1}{2(d - w - 1)} \leq \epsilon_0 \Rightarrow d > \ln \frac{4}{\delta} \frac{1}{2 \epsilon_d} + w + 1. \quad (29)$$

Here (*) follows because $W_d(t) = d \leq W_m(t)$. This shows that whenever $W_d(t) = d \leq W_m(t)$ and the windows are loaded with non-coherent samples, error will be detected with probability $1 - \delta$.

(Part (b)) Note that for any time $t$, the distribution will be declared changed if $\|\hat{\pi}_i^d(t) - \pi_m^m(t)\|_{TV} > \epsilon_d$. Choose $\epsilon_d = 2\epsilon_1$. Similar to the above, we have:

$$\Pr \{ \|\hat{\pi}_i^d(t) - \pi_m^m(t)\|_{TV} \geq \epsilon_d \} \leq \Pr \{ \|\hat{\pi}_i^d(t) - \pi_m^m(t)\|_{TV} \geq \epsilon_d + \epsilon_1 \} \leq \Pr \{ \|\hat{\pi}_i^d(t) - \pi_m^m(t)\|_{TV} \geq \epsilon_d/2 \} + \Pr \{ \|\pi_m^m(t) - \pi_m^m(t)\|_{TV} \geq (1 - \alpha)\epsilon_d/2 \}. \quad (30)$$

Using the same argument as in (25), (26) and (27), we get:

$$\Pr \{ \|\hat{\pi}_i^d(t) - \pi_m^m(t)\|_{TV} \geq \epsilon_d \} \leq \delta.$$  

This shows that step (i) declares change with probability $\delta$.

Next we show that step (ii) does not declare a distribution change with high probability. To do so, we use Theorem 5 with $m = 2\log(T_i)/\sqrt{T_i}$ to have that, when $W_m(t) \geq T_i$,

$$\Pr \{ \|\pi_m^m(t) - \pi^m\| > \frac{2\log(T_i)}{\sqrt{T_i}} \} \leq e^{-2\log(T_i)^2} = T_i^{-2\log(T_i)}.$$  

Using the union bound, we get

$$\Pr \{ \|\pi_m^m(t) - \pi^m\| > \frac{2M\log(T_i)}{\sqrt{T_i}} \} \leq MT_i^{-2\log(T_i)}. \quad (31)$$

Thus, part (b) follows from the union bound over $k$. □