# Price of Anarchy of Wireless Congestion Games

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**Abstract**

In this paper, we consider a resource allocation game with heterogenous players competing for limited resources. We model this as a singleton congestion game, where the share of each player is a decreasing function of the number of players selecting the same resource. In particular, we consider player-specific payoffs that depend not only on the shares of resource, but also on different preference constants. We study the price of anarchy (PoA) for four families of this congestion game: identical, player-specific symmetric, resource-specific symmetric, and asymmetric games. We characterize the exact worst-case PoA in terms of the number of players and resources. By comparing the values of PoA for different games, we show that performance loss increases with the heterogeneity of games (i.e., the identical game has a better PoA in general). From a system design point of view, we identify the worst-case Nash equilibrium, where all players are competing for a single resource in asymmetric games. We present an application of this class of congestion games to spectrum sharing in cognitive radio network with two medium access control schemes.

## I. Introduction

Congestion game has long been a useful tool in modeling problems in job scheduling and selfish routing (e.g., [1]–[3]). They capture the negative congestion effect through a cost or latency function which is increasing in the number of players sharing the same resource. There exists a large volume of literature studying the existence, convergence, and efficiency of Nash equilibrium in these game models (e.g., [1], [4], [5]).

Rather than cost or delay, in this paper we study the congestion effect by focusing on the share of common resource obtained by each player. When multiple players share the same resource, each player's share is a decreasing function in the number of players. Furthermore, we consider the case where players are heterogeneous, each modeled by a player-specific payoff as the product of the share of resource and a multiplicative user-specific preference constant. Each player aims to maximize his own payoff by selecting the right resource. We illustrate this with two motivating examples:

Example *1 (Internet service subscription):* A user can select one out of many Internet Service Providers (ISPs) to obtain Internet service. Assuming each ISP has a fixed amount of total bandwidth and allocates resources fairly among its subscribers. The more users subscribing to an ISP, the smaller a share of bandwidth obtained by each user. Depending on individual needs, prices, and Quality of Service (QoS) requirements, different users may have different evaluations on the amount of bandwidth received from different providers. Hence, a user's utility is a product of both his share of resources and individual preference. Each user decides which ISP to subscribe to maximize his utility.

Example *2 (Spectrum sharing):* In a wireless cognitive radio network, secondary unlicensed users sense the spectrum to locate and exploit "spectrum holes", or temporarily available spectrum at different times, locations, and frequencies that are not used by primary licensed users. Due to various hardware constraints, each secondary user might only be able to sense and transmit in one channel at a time. A user's chance of successfully accessing a channel decreases as the number of secondary users sensing the same channel increases. Furthermore, the data rates perceived by different users vary based on their respective channel gains and geographical locations. In this case, the expected data rate of a user is the product of the user-specific data rate and his probability of successful transmission. The goal of a secondary user is to select a channel to sense in order to maximize his expected data rate.

In this paper, we focus on a special class of congestion games called the *singleton congestion games*, where each player is allowed to select only one resource from the set. This captures the fact that choices are sometimes restricted, and that splitting over multiple resources may not be possible. Furthermore, the payoff from using multiple different resources are not necessarily additive in general.

We focus on the solution concept of Nash equilibrium, where each player has no incentive

TABLE I: Price of anarchy for singleton congestion games.

| | Identical | Player-specific symmetric | Resource-specific symmetric | Asymmetric |
|---|---|---|---|---|
| Special allocation functions | [14], [15] | | [12], [14], [15], This paper | |
| General allocation functions | This paper | This paper | | This paper |

to unilaterally change his current choice of resource. However, a Nash equilibrium does not achieve social optimality in general. We will evaluate this inefficiency by studying the price of anarchy (PoA). We study several types of singleton congestion games and characterize the impact of players' selfish behavior on the social welfare. As a concrete example, we will apply singleton congestion games to the study of wireless spectrum sharing in cognitive radio networks. In particular, we will use our results on PoA to illustrate the difference between two different medium access control schemes in this context. To the best of our knowledge, this paper is the first to study the PoA for general singleton congestion games with multiplicative user-specific preference constants.

### *A. Related Work*

Reference [1] was among the earliest work that modeled road traffic and firm productions using the congestion game model. Extensive research has been done on the application of congestion games to selfish routing problems (e.g., [2] and [6]). In these models, each selfish player aims to minimize his latency, which is a non-decreasing function of the load on the path.

Reference [8] was the first one that considered congestion games with player-specific payoffs. The authors showed that a pure Nash equilibrium always exists when players can select only one resource in their strategy. Since games with player-specific payoffs are in general difficult to analyze, some recent work started to consider a special class of congestion games with player-specific constants. It is shown in [9] and [10] that pure Nash equilibria exist for this type of singleton games.

The study of inefficiency in system performance due to selfish behaviors of players was initiated by [11]. The bounds of PoA for congestion games with linear and polynomial cost functions were proven in [5] and [12]. Recent work such as [13] gave the exact PoA for a class of congestion games with cost functions restricted to a fixed set. Similar results on the bounds or the computation of PoA of singleton congestions games are given for some families of games as listed in Table I. We divided congestion games into four families (identical, player-specific symmetric, resource-specific symmetric, and asymmetric games) based on the different constraints imposed on the preference constants (details are given in Section III). In [12], [14], and [15], the authors considered congestion games with polynomial latency functions in load balancing. Players choosing the same path experience the same latency and hence the model fall into the two families of identical and resource-specific symmetric games. In this paper, we compute the worst-case PoA in different families of singleton congestion games as listed in Table I.

Spectrum sharing is a main motivating application for the singleton congestion games studied in this paper. Cognitive radio has recently emerged as a promising technology to alleviate the problem of under-utilized spectrum resources (e.g., [16], [17]). There is a large volume of literature on spectrum sharing in cognitive radio networks (e.g., [18], [19]). In [7], congestion game is used to model wireless spectrum sharing games where spatial reuse of wireless channels is taken into account. In this paper, we aim to model spectrum sharing games in cognitive radio with congestion games and study the inefficiency of the Nash equilibria.

### B. Contributions

The main results and contributions of this paper are summarized as follows:

- *Price of anarchy analysis of general singleton congestion games with preference constants.* We study a general model of $K$ players, $M$ resources, and a general decreasing allocation function $r(n)$, and multiplicative preference constants. Each player can select exactly one resource only. Such single choice makes the analysis of PoA different (and often more challenging) from the traditional congestion games that allow the usage of multiple resources. We compute the exact worst-case PoA[1] as summarized in Table II. The detailed notations are introduced in Section II.
- *Insight on better system design.* As a by-product of the proofs on PoA, we also identify the worst-case Nash equilibrium. We show that all players selecting the same resource leads to

[1]The results in resource-specific symmetric games are restricted to a special allocation function $r(n) = \frac{1}{n}$ as discussed in Section VI-B.

TABLE II: Worst-case PoA for the different families of games in terms of $K$, $M$, and general $r(n)$ functions.

| | $K \leq M$ | $K > M$ |
|---|---|---|
| Identical | 1 | $\frac{(K \mod M)f(\lfloor\frac{K}{M}\rfloor+1)+[M-(K \mod M)]f(\lfloor\frac{K}{M}\rfloor)}{(M-1)+f(K-M+1)}$ |
| Player-specific symmetric | 1 | $r(\lfloor\frac{K}{M}\rfloor)$ [2]or $r(\lfloor\frac{K}{M}\rfloor+1)$ |
| Asymmetric | $r(K)$ | $r(K)$ |

the worst possible outcome in asymmetric games. In addition, we evaluate the worst-case PoA in different families of games when the number of players varies. These results shed light on how to design systems with smaller efficiency loss by controlling the number of players competing for resources or the level of heterogeneity among resources and players.

- *Application of congestion games in cognitive radio networks.* We apply the general model of singleton congestion games to wireless cognitive radio networks with two medium access control schemes: *uniform MAC* and *slotted Aloha*. The worst-case PoA obtained under slotted Aloha is in general smaller (worse) than that under the uniform MAC in all four families of games, due to the resource waste in slotted Aloha. Moreover, the worst-case PoA decreases in the order of identical games, symmetric games, and asymmetric games.

## II. Game Model

We model the competition for several resources among a group of heterogenous players as a congestion game. The key notations of this paper are listed in Table V. Consider the game tuple $(\mathcal{K}, \mathcal{M}, (\Sigma_k)_{k\in\mathcal{K}}, (\pi_m^k)_{k\in\mathcal{K},m\in\mathcal{M}})$, where $\mathcal{K} = \{1, ..., K\}$ is the set of players, $\mathcal{M} = \{1, ..., M\}$ the set of resources, and $\Sigma_k$ the set of pure strategies for player $k$. Since all players have the same set of resources for selection and the player's strategy consists of a single resource only, we have $\Sigma_k = \mathcal{M}$ for all $k$. Finally, $\pi_m^k = R_m^k r(n_m)$ is the payoff function of player $k$ for selecting resource $m$. The preference constant $R_m^k$ denotes the preference of player $k$ for selecting resource $m$. The resource allocation function $r(n_m)$ is in the number of players selecting resource $m$, $n_m$; this is the share of resource $m$ each of the $n_m$ players gets. We will assume that the allocation of

[2]The PoA is $r(\lfloor\frac{K}{M}\rfloor)$ if $(K \mod M) = 0$ or $r(\lfloor\frac{K}{M}\rfloor+1)$ otherwise.

shares to different players selecting the same resource is the same due to fairness consideration, and thus this is a common function to all players. In addition, we assume the function $r(n_m)$ satisfies the following properties:

- $r(1) = 1$, i.e., a single player can fully utilize a single resource.
- $r(n_m)$ is a decreasing function of $n_m$, which captures the fact that more players using the same resource leads to a smaller allocation per player.
- $n_m r(n_m) \leq 1$, since the sum of resources shared among players cannot exceed 1. The strict inequality case refers to the scenario where resource waste happens due to congestion (e.g., collision in wireless communications).

Each player $k$ selects a single resource that maximizes his own payoff, i.e., $\max_{m \in \Sigma_k} \pi_m^k$. A pure strategy profile is given by $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_K)$, where $\sigma_k \in \Sigma_k$ denotes the resource that player $k$ selects. The set of strategy profiles is denoted by $\Pi = \Sigma_1 \times \Sigma_2 \times ... \times \Sigma_K$. The profile $\boldsymbol{\sigma}$ is a *Nash equilibrium* if and only if no player can improve his payoff by deviating unilaterally, i.e., for each player $k \in \mathcal{K}$,

$$R_{\sigma_k}^k r(n_{\sigma_k}) \geq R_j^k r(n_j + 1), \ \ \forall j \in \mathcal{M} \text{ and } j \neq \sigma_k.$$

We denote by $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, ..., n_M)$ the *congestion vector* corresponding to the strategy profile $\boldsymbol{\sigma}$.

To facilitate the discussion, we will use the term $(K, M)$-*game* to represent a game with $K$ players and $M$ resources.

## III. Price of Anarchy (PoA)

Our singleton congestion game with preference constants belongs to the class of congestion games which always has a pure Nash equilibrium. This can be proved by showing that there exists an ordinal potential function. The function increases as players update their strategies myopically and is upper-bounded, and hence a pure Nash equilibrium always exists. Meanwhile, it is common to have multiple Nash equilibria in a singleton congestion game.

A Nash equilibrium is the stable outcome of distributed selfish behaviors by all players. It is not difficult to imagine that such behavior often leads to the loss of social welfare. Given at least one Nash equilibrium exists in our game, a natural question to ask is how far the Nash equilibrium is from the social optimum. One metric to quantify this is the price of anarchy (PoA), which is the focus of this paper.

Before defining PoA, let us define the social optimum and the efficiency ratio. Denote the total payoff received by all players at a given Nash equilibrium $\boldsymbol{\sigma}$ as

$$SUM(\boldsymbol{\sigma}) = \sum_{k \in \mathcal{K}} \pi^k_{\sigma_k} = \sum_{k \in \mathcal{K}} R^k_{\sigma_k} r(n_{\sigma_k}).$$

*Definition 1:* The *social optimum opt* of the game is the maximum total payoff received by all $K$ players maximized over all strategy profiles[3],

$$opt = \max_{\boldsymbol{\sigma} \in \Pi} SUM(\boldsymbol{\sigma}).$$

Any strategy profile $\boldsymbol{\sigma}$ that leads to a social optimum is called a socially optimal solution. Similar to the Nash equilibria, there can be multiple $\boldsymbol{\sigma}$'s (i.e., player-resource assignments) that lead to the same social optimum.

*Definition 2:* The *efficiency ratio* of a Nash equilibrium $\boldsymbol{\sigma}$ is the ratio between the total payoff received at that equilibrium and the social optimum,

$$ER(\boldsymbol{\sigma}) = \frac{SUM(\boldsymbol{\sigma})}{opt}.$$

*Definition 3:* The *price of anarchy (PoA)* of a game is the worst-case efficiency ratio among all pure strategy Nash equilibria,

$$PoA(K, M, \boldsymbol{R}) = \min_{\boldsymbol{\sigma} \in \Pi} \frac{SUM(\boldsymbol{\sigma})}{opt} = \min_{\boldsymbol{\sigma} \in \Pi} ER(\boldsymbol{\sigma}). \tag{1}$$

Here $\boldsymbol{R} = \{R^k_m, \forall k \in \mathcal{K}, \forall m \in \mathcal{M}\}$ denotes the preference constants of all players. There is a slight difference between the PoA defined here and the one defined in many prior works (e.g., [11]). Since we are maximizing payoff instead of minimizing cost, the social optimum is the largest value among all total payoffs. Thus, PoA defined here never exceeds 1, and the worst Nash equilibrium achieves the smallest efficiency ratio and thus the PoA.

In general, PoA is defined for a particular game with all parameters specified. For our model, these parameters include the number of players ($K$), number of resources ($M$), and the preference constants ($R^k_m$'s). The PoA is the worst-case ratio among all Nash equilibria and the social optimum in such a game. We can further extend the concept of PoA from a game to a *family* of

[3]Such social optimum can be achieved, for example, through a centralized scheduler who tells each player which resource to select. The congestion cannot be completely avoided even at a socially optimal solution as long as $K > M$.

game. In particular, we are interested in the smallest value of PoA among all possible choices of preference constants. This is referred to as the *worst-case PoA* defined below.

*Definition 4:* The *worst-case PoA* of a family of games is the minimum one over all possible preference constants, i.e., $\min_{\boldsymbol{R}} PoA(K, M, \boldsymbol{R})$.

*Definition 5:* The *universal worst-case PoA* of a family of games is $\min_{K,M,\boldsymbol{R}} PoA(K, M, \boldsymbol{R})$.

As we will see, we can often compute the worst-case PoA without having to consider all possible combinations of preference constants. To facilitate the study, we classify the congestion game into several families depending on the heterogeneity of players and resources.

- *Identical game*: all resources are the same for all players, i.e., $R_m^k = R$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$.
- *Player-specific symmetric game*: different players have different preferences for resources, but each player has the same preference constant for different resources, i.e., $R_m^k = R^k$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$.
- *Resource-specific symmetric game*: all players have the same preference constant for the same resource, but have different preferences for different resources, i.e., $R_m^k = R_m$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$.
- *Asymmetric game*: this is the most general case where each resource is different for different players, i.e., $R_m^k$ can be different for each $k \in \mathcal{K}$ and $m \in \mathcal{M}$.

The four families of games have an increasing level of complexity but they are not mutually exclusive. The identical game is a special case of the symmetric game, which in turn is a special case of the asymmetric game. Note however, that they are not subsets of each other in terms of the worst-case PoA as different families impose different constraints on the selections of preference constants. For example, identical games are not the worst possible case for symmetric games. Furthermore, the families of player-specific and resource-specific symmetric games are in general more difficult to analyze due to the constraints on the preference constants.

To begin with, we consider the trivial case of $M = 1$. When there is one type of resource, all players have no choice but to select the only resource. The total payoff at Nash equilibrium is the same as the social optimum, hence PoA=1. In the rest of the paper, we will look at the nontrivial case of $M \geq 2$. Due to page limit, we only provide proof sketches for most of the results. For details, please see the online technical report [20].

## IV. PoA Analysis of Identical Games

We first start with the family of identical games, which is the simplest to analyze. With the condition of $R_m^k = R$, we can identify both the Nash equilibria and social optimum explicitly.

Consider $K$ players in an identical game. Denote $\sigma_k$ to be the resource selected by player $k$. Then, the total payoff of all players is

$$\sum_{k\in\mathcal{K}} \pi_{\sigma_k}^k = \sum_{k\in\mathcal{K}} R_{\sigma_k}^k r(n_{\sigma_k}) = \sum_{k\in\mathcal{K}} Rr(n_{\sigma_k}) = R\sum_{m\in\mathcal{M}} n_m r(n_m).$$

By definition, all players have the same preference constant for all resources, i.e., $R_m^k = R$ for all player $k \in \mathcal{K}$ and all resource $m \in \mathcal{M}$. This results in the second equality where $R_{\sigma_k}^k = R$. The last equality results from a change of summation from players to resources. We now define a new sharing function associated with the total share of the $n$ players selecting the same resource,

$$f(n) = \begin{cases} 0, & \text{if } n = 0. \\ nr(n), & \text{if } n \geq 1. \end{cases}$$

Therefore, the total payoff received by $K$ players can be written as a sum of payoffs generated by each resource, i.e., $R\sum_{m\in\mathcal{M}} f(n_m)$ where $\sum_{m\in\mathcal{M}} n_m = K$.

In some scenarios, congestions in resources would reduce the total payoff of players when the resource is not completely shared. In addition, the reduction would be less significant when there already exists a large number of players. This motivates us to make the following assumption.

*Assumption 1:* The function $f(n)$ is convex and non-increasing in $n$, i.e., $f'(n) \leq 0$ and $f''(n) \geq 0$.

Assumption 1 implies that $f'(n) = nr'(n) + r(n) \leq 0$ and $f''(n) = nr''(n) + 2r'(n) \geq 0$, which implies $r'(n) < 0$ and $r''(n) > 0$. In other words, $r(n)$ is a decreasing and convex function under Assumption 1. This assumption is naturally satisfied in many applications. For example, in a medium access control game in cognitive radio networks, the collisions of secondary transmissions reduce the probability of successful transmission. This results in a lower efficiency and hence a smaller value of $f(n)$ with an increasing number of players.

*Lemma 1:* For an identical ($K$,$M$)-game with preference constant $R$, the total payoff at a socially optimal solution is

$$\mathit{opt} = \begin{cases} RK, & \text{if } K \leq M. \\ R(M-1) + Rf(K-M+1), & \text{if } K > M. \end{cases}$$

*Proof:* If the number of players is no larger than the number of resources (i.e., $K \leq M$), then each player selecting a different resource leads to the social optimum $RK$. This is because $nr(n) \leq r(1) = 1$, thus preventing players from sharing resources. When there are more players than resources (i.e., $K > M$), it is optimal to add the additional $K - M$ players to a single resource. This is because due to the non-increasing and convexity properties of $f(n)$, we have $f(n_1) + f(n_2) \leq f(1) + f(n_1 + n_2 - 1)$ for any $n_1, n_2 \geq 1$. Hence, we have $opt = R(M-1) + Rf(K - M + 1)$ for $K > M$. ■

In the special case of $f(n) = 1$ for any $n$, i.e., no matter how many players share a resource, the shares sum up to the same total amount, the total payoff is thus independent of the number of players selecting it, as long as it is positive. The social optimum in this case is $R \min(K, M)$.

We next calculate the total payoff at different Nash equilibria[4] and hence the PoA.

*Theorem 1:* For an identical $(K, M)$-game[5],

$$PoA = \begin{cases} 1, & \text{if } K \leq M. \\ \frac{(K \mod M) f(\lfloor \frac{K}{M} \rfloor + 1) + [M - (K \mod M)] f(\lfloor \frac{K}{M} \rfloor)}{(M-1) + f(K-M+1)}, & \text{if } K > M. \end{cases}$$

*Proof:* Consider any Nash equilibrium $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_K)$ with the congestion vector $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, ..., n_M)$, where $\sum_{m \in \mathcal{M}} n_m = K$. For each player $k$, $Rr(n_{\sigma_k}) \geq Rr(n_j + 1), \ \ \forall k \in \mathcal{K}, \forall j \in \mathcal{M} \neq \sigma_k$. It is a player's best response to select the least congested resource given other players' fixed choices. This results in an "even" distribution of players on all resources.

To be more precise, we prove by contradiction that the difference of number of players on any two resources is no greater than 1 in any Nash equilibrium, i.e., $|n_i - n_j| \leq 1, \ \ i, j \in \mathcal{M}$. Suppose there exists a pair of resources $i$ and $j$ where $n_i = n_j + t$ and $t > 1$. According to the definition of Nash equilibrium, $r(n_i) \geq r(n_j + 1)$. Otherwise, at least one player selecting resource $i$ will switch to resource $j$. However, since $n_i = n_j + t > n_j + 1$ and $r'(n) < 0$, we have $r(n_i) < r(n_j + 1)$. This leads to a contradiction.

Therefore, we can identify the congestion vector in different Nash equilibria.

- For the case of $K \leq M$:

  Since number of players is no greater than number of resources, players always prefer an

[4]There in general exist multiple Nash equilibria for different "pairing" of players and resources. However, the total payoff at all Nash equilibria is the same in a symmetric game and the identities of players are not important.

[5]The modulo operation ($a \mod n$) is the remainder on division of $a$ by $n$, and the floor function $\lfloor a \rfloor$ is the largest integer not greater than $a$.

unused resource to a congested resource. Therefore, each player selects a different resource in all Nash equilibria. Though the players-resources assignment may vary in different Nash equilibria, the total payoff by the group of players is always the same, i.e., $SUM(\boldsymbol{\sigma}) = RK$.

- For the case of $K > M$:

  We showed in the above that the difference of number of players on any two resources is no greater than 1 in any Nash equilibria. If $K$ is divisible by $M$, every resource is selected by the same number of players, $\frac{K}{M}$. Otherwise, the remainder of players select a different resource to guarantee an even distribution of players. A number of $(K \mod M)$ resources are selected by $\lfloor \frac{K}{M} \rfloor + 1$ players; while the remaining $M - (K \mod M)$ resources are selected by $\lfloor \frac{K}{M} \rfloor$ players. The total payoff at Nash equilibrium $\boldsymbol{\sigma}$, $SUM(\boldsymbol{\sigma}) = R(K \mod M) f(\lfloor \frac{K}{M} \rfloor + 1) + R[M - (K \mod M)] f(\lfloor \frac{K}{M} \rfloor)$.

The above gives the unique value of $SUM(\boldsymbol{\sigma})$ for all possible Nash equilibria. Combining with the results in Lemma 1 and applying the definition of PoA, we obtain the PoA for identical games. ■

Theorem 1 implies that the PoA is independent of the preference constant $R$, and hence is the same as the worst-case PoA ($\min_{\boldsymbol{R}} PoA(K, M, \boldsymbol{R})$).

*Remark 1: (Asymptotic PoA)* Let $K = tM + y$, where $t$ is a positive integer and $0 \leq y < M$, then Theorem 1 can be written as $PoA = \frac{yf(t+1) + (M-y)f(t)}{(M-1) + f((t-1)M + y + 1)}$. If $M$ is fixed and $t \to \infty$, then $PoA = \lim_{t \to \infty} \frac{Mf(t)}{M - 1 + f(t)}$. When the number of players is significantly larger than resources, the PoA is getting smaller and approaches $\lim_{t \to \infty} f(t)$ eventually.

## V. PoA Analysis of Asymmetric Games

In asymmetric games, the preference constants are different for different players on different resources. The social optimum is difficult to compute in this case. However, it turns out that we can compute the PoA by exploiting the properties of the social optimum without explicitly computing the latter.

*Lemma 2:* For an asymmetric $(K, M)$-game, $\min(K, M)$ of resources are selected at a socially optimal solution.

*Proof:* We first consider a simple case with two resources, where one is selected by more than one player and the other remains unused. We show by contradiction that the total payoff can always be improved by switching a player from a congested resource to an unused resource.

Suppose $K$ players selecting the same resource is a socially optimal solution. Without loss of generality, we assume all players are selecting resource 1. By labeling the players in descending order of their preference constants for resource 1, we have $R_1^1 \geq R_1^2 \geq \cdots \geq R_1^K$. By assumption, the optimal total payoff in this case is $\sum_{k \in \mathcal{K}} R_1^k r(K)$.

Now, we consider another scenario where all players except player $K$ choose resource 1. The new total payoff

$$
\begin{aligned}
&= \sum_{k \in \mathcal{K} \backslash \{K\}} R_1^k r(K-1) + R_2^K = \frac{r(K-1)}{r(K)} \sum_{k \in \mathcal{K} \backslash \{K\}} R_1^k r(K) + R_2^K \\
&= \sum_{k \in \mathcal{K} \backslash \{K\}} R_1^k r(K) + \left[\frac{r(K-1)}{r(K)} - 1\right] \sum_{k \in \mathcal{K} \backslash \{K\}} R_1^k r(K) + R_2^K \\
&= \sum_{k \in \mathcal{K}} R_1^k r(K) - R_1^K r(K) + \left[\frac{r(K-1)}{r(K)} - 1\right] \sum_{k \in \mathcal{K} \backslash \{K\}} R_1^k r(K) + R_2^K \\
&= \sum_{k \in \mathcal{K}} R_1^k r(K) - \frac{1}{K-1} \sum_{k \in \mathcal{K} \backslash \{K\}} R_1^K r(K) + \left[\frac{r(K-1)}{r(K)} - 1\right] \sum_{k \in \mathcal{K} \backslash \{K\}} R_1^k r(K) + R_2^K \\
&> \sum_{k \in \mathcal{K}} R_1^k r(K) - \left[\frac{r(K-1)}{r(K)} - 1\right] \sum_{k \in \mathcal{K} \backslash \{K\}} R_1^K r(K) + \left[\frac{r(K-1)}{r(K)} - 1\right] \sum_{k \in \mathcal{K} \backslash \{K\}} R_1^k r(K) + R_2^K \\
&= \sum_{k \in \mathcal{K}} R_1^k r(K) + \left[\frac{r(K-1)}{r(K)} - 1\right] r(K) \sum_{k \in \mathcal{K} \backslash \{K\}} (R_1^k - R_1^K) + R_2^K > \sum_{k \in \mathcal{K}} R_1^k r(K).
\end{aligned}
$$

The first inequality is obtained from the assumption that $f'(n) < 0$ and is verified as follow: $f(K-1) > f(K) \Rightarrow (K-1)r(K-1) > Kr(K) \Rightarrow \frac{r(K-1)}{r(K)} > \frac{K}{K-1} \Rightarrow \frac{r(K-1)}{r(K)} - 1 > \frac{1}{K-1}$.

The last inequality is due to the facts that (i) $R_1^k \geq R_1^K$ for all players $k$ and $R_2^K \geq 0$; (ii) $r(K-1) \geq r(K)$ since $r(n)$ is a decreasing function. Hence, the deviation of player $K$ leads to a better payoff and thus a contradiction. We concluded that the optimum is obtained only when both resources are selected.

Below we show that exactly $K$ resources are selected at any socially optimal solution for $K \leq M$.

Suppose $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_K)$ is a strategy profile that leads to the optimum. By considering its corresponding congestion vector $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, ..., n_M)$, the $M$ resources can be divided into 3 sets:

- $\mathcal{S}_0 = \{m \in \mathcal{M} : n_m = 0\}$ denotes the set of unused resources
- $\mathcal{S}_1 = \{m \in \mathcal{M} : n_m = 1\}$ denotes the set of resources selected by 1 player

- $\mathcal{S}_2 = \{m \in \mathcal{M} : n_m > 1\}$ denotes the set of resources selected by more than 1 player

If there exists a resource $i \in \mathcal{S}_2$, then there must also exist a resource $j \in \mathcal{S}_0$. This is because the number of players is no more than the number of resources. From the earlier analysis on two resources, we know that deviation of a player from resource $i$ to resource $j$ always improves the total payoff. Therefore $\boldsymbol{\sigma}$ is not a socially optimal solution. This argument can be successively applied until all resources are either unused or selected by one player, such that $\mathcal{S}_2 = \phi$. We then conclude that the optimum is obtained when exactly $K$ resources are selected. In this case, each player selects a different resource in the social optimum.

Using a similar argument, we can also show by contradiction that all $M$ resources are selected at any socially optimal solution in the case of $K > M$. ■

Lemma 2 characterizes the properties of all socially optimal solutions without specifying the player-resource associations. Such partial characterization turns out to be sufficient for the subsequent analysis of PoA.

We will compute the exact worst-case PoA in two steps. We first give a lower-bound for the efficiency ratio, and then show that the bound is achievable.

*Lemma 3:* For an asymmetric $(K, M)$-game with $M > 1$, the lower-bound of efficiency ratio is $r(K)$.

*Proof:* We consider two cases separately: $K \leq M$ and $K > M$.

*For the case of $K > M$:*

For an equilibrium $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_K)$, we have $R^k_{\sigma_k} r(n_{\sigma_k}) \geq R^k_j r(n_j + 1)$ for all $k \in \mathcal{K}$ and $j \in \mathcal{M}$ and $j \neq \sigma_k$. Here $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, n_2, ..., n_M)$ is the corresponding congestion vector of $\boldsymbol{\sigma}$. We will also denote $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_K)$ as the strategy profile played by the players at a socially optimal solution, and $\boldsymbol{q} = (q_1, q_2, ..., q_M)$ be the corresponding congestion vector.

By using the inequality $\frac{\sum_i a_i}{\sum_i b_i} \geq \min_i \frac{a_i}{b_i}$, we have

$$\mathrm{ER}(\sigma) = \frac{\sum_{k \in \mathcal{K}} R^k_{\sigma_k} r(n_{\sigma_k})}{\sum_{k \in \mathcal{K}} R^k_{\omega_k} r(q_{\omega_k})} \geq \min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_{\omega_k} r(q_{\omega_k})}.$$

Assume $\bar{k} = \arg\min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_{\omega_k} r(q_{\omega_k})}$, we have two possible scenarios:

- If $\sigma_{\bar{k}} = \omega_{\bar{k}}$, then

$$\min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_{\omega_k} r(q_{\omega_k})} = \frac{R^{\bar{k}}_{\omega_{\bar{k}}} r(n_{\omega_{\bar{k}}})}{R^{\bar{k}}_{\omega_{\bar{k}}} r(q_{\omega_{\bar{k}}})} = \frac{r(n_{\omega_{\bar{k}}})}{r(q_{\omega_{\bar{k}}})} \geq \frac{r(K)}{r(1)} = r(K).$$

The last inequality is due to the fact that $r(n)$ is a decreasing function. Therefore, $n_{\sigma_k}$ taking its maximum value of $K$ while $q_{\omega_k}$ taking the minimum value of 1 leads to the lower-bound. Here $K$ players select resource $\omega_k$ in the Nash equilibrium while only player $\bar{k}$ selects the same resource at the socially optimal solution.

- If $\sigma_{\bar{k}} \neq \omega_{\bar{k}}$, then

$$\min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_{\omega_k} r(q_{\omega_k})} = \frac{R^{\bar{k}}_{\sigma_{\bar{k}}} r(n_{\sigma_{\bar{k}}})}{R^{\bar{k}}_{\omega_{\bar{k}}} r(q_{\omega_{\bar{k}}})} \geq \frac{R^{\bar{k}}_{\omega_{\bar{k}}} r(n_{\omega_{\bar{k}}}+1)}{R^{\bar{k}}_{\omega_{\bar{k}}} r(q_{\omega_{\bar{k}}})} = \frac{r(n_{\omega_{\bar{k}}}+1)}{r(q_{\omega_{\bar{k}}})} \geq \frac{r(K)}{r(1)} = r(K).$$

The first inequality is due to player $\bar{k}$'s best response at the Nash equilibrium where $R^{\bar{k}}_{\sigma_{\bar{k}}} r(n_{\sigma_{\bar{k}}}) \geq R^{\bar{k}}_{\omega_{\bar{k}}} r(n_{\omega_{\bar{k}}}+1)$. The lower-bound is obtained when player $\bar{k}$ switches his strategy at the Nash equilibrium to that of the social optimum. The last inequality follows a similar argument as before.

The efficiency ratios in both cases are lower-bounded by $r(K)$.

*For the case of $K \leq M$:*

From Lemma 2, we know that all $K$ players should be selecting $K$ different resources at a socially optimal solution. Without loss of generality, label the resources and players so that the optimum is $\sum_{k\in\mathcal{K}} R^k_k$.

Denote the strategy played by player $k$ as $\sigma_k$ and the strategy profile by $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_K)$. The congestion vector of $\boldsymbol{\sigma}$ is given by $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, n_2, ..., n_M)$. The strategy profile $\boldsymbol{\sigma}$ is a Nash equilibrium if and only if $R^k_{\sigma_k} r(n_{\sigma_k}) \geq R^k_j r(n_j+1), \quad \forall k \in \mathcal{K}, \forall j \in \mathcal{M} \neq \sigma_k$.

By using the inequality $\frac{\sum_i a_i}{\sum_i b_i} \geq \min_i \frac{a_i}{b_i}$, we have

$$\text{ER}(\sigma) = \frac{\sum_{k\in\mathcal{K}} R^k_{\sigma_k} r(n_{\sigma_k})}{\sum_{k\in\mathcal{K}} R^k_k} \geq \min_k \frac{R_{k\sigma_k} r(n_{\sigma_k})}{R^k_k}.$$

Assume $\bar{k} = \arg\min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_k}$, we have two possible scenarios:

- If $\sigma_{\bar{k}} = \bar{k}$, then

$$\min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_k} = \frac{R^{\bar{k}}_{\bar{k}} r(n_{\bar{k})}}{R^{\bar{k}}_{\bar{k}}} = r(n_{\bar{k}}) \geq r(K).$$

In this case, the strategies played by player $\bar{k}$ in both the Nash equilibrium and a socially optimal solution are the same. In addition, $n_{\bar{k}}$ takes its maximum value of $K$ implies that all players select resource $\bar{k}$ in the Nash equilibrium.

- If $\sigma_{\bar{k}} \neq \bar{k}$, then

$$\min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_k} = \frac{R^{\bar{k}}_{\sigma_{\bar{k}}} r(n_{\sigma_{\bar{k}}})}{R^{\bar{k}}_{\bar{k}}}$$
$$\geq \frac{R^{\bar{k}}_{\bar{k}} r(n_{\bar{k}}+1)}{R^{\bar{k}}_{\bar{k}}} = r(n_{\bar{k}}+1) \geq r(K).$$

Here, the strategies played by player $\bar{k}$ in both Nash equilibrium and a socially optimal solution are different. The first inequality is due to player $\bar{k}$'s best response at a Nash equilibrium. Player $\bar{k}$ selects resource $\sigma_{\bar{k}}$ in Nash equilibrium because $R^{\bar{k}}_{\sigma_{\bar{k}}} r(n_{\sigma_{\bar{k}}}) \geq R^{\bar{k}}_{\bar{k}} r(n_{\bar{k}}+1)$. Therefore, there can be at most $K-1$ players selecting resource $\bar{k}$ in that Nash equilibrium.

The efficiency ratios in both cases are lower-bounded by $r(K)$. ■

*Lemma 4:* For any $\delta > 0$, there exists an asymmetric ($K$,$M$)-game with $M > 1$ that has an efficiency ratio $r(K)+\delta$.

*Proof:* We now consider an asymmetric ($K$,$M$)-game with $M > 1$ where the preference constants, $R^k_m$ are shown in Table IV. The $\epsilon$ in the table is specially chosen as a function of $\delta$ for the different cases of $K \leq M$ and $K > M$.

*For the case of $K \leq M$:* The value of $\epsilon$ in the table is chosen as $\frac{\delta R}{[1-r(K)-\delta](K-1)r(K)}$.

With the condition $Rr(K) > \epsilon$, the social optimum is achieved when player $k$ is selecting resource $k$ for all $k \in \mathcal{K}$. In that case, the total payoff would be $R+(K-1)\epsilon r(K)$ at the socially optimal solution. On the other hand, we can identify one Nash equilibrium $\boldsymbol{\sigma}$ where all players select resource 1. The total payoff at $\boldsymbol{\sigma}$ is $Rr(K)+(K-1)\epsilon r(K)$. Therefore, we have

$$\text{ER}(\boldsymbol{\sigma}) = \frac{Rr(K)+(K-1)\epsilon r(K)}{R+(K-1)\epsilon r(K)}$$
$$= r(K) + \frac{[1-r(K)](K-1)\epsilon r(K)}{R+(K-1)\epsilon r(K)}$$
$$= r(K)+\delta,$$

Furthermore, we show that $\epsilon$ is an increasing function of $\delta$.

$$\epsilon = \frac{\delta R}{[1-r(K)-\delta](K-1)r(K)}$$
$$\frac{\partial \epsilon}{\partial \delta} = \frac{[1-r(K)-\delta](K-1)r(K)R - \delta R[-(K-1)r(K)]}{\{[1-r(K)-\delta](K-1)r(K)\}^2}$$
$$= \frac{[1-r(K)](K-1)r(K)R}{\{[1-r(K)-\delta](K-1)r(K)\}^2} > 0$$

TABLE III: Example of An Asymmetric Game with Efficiency Ratio $r(K)+\delta$

| player \ resource | 1 | 2 | $\cdots$ | $M$ |
|---|---|---|---|---|
| 1 | $R$ | $Rr(K)$ | $\cdots$ | $Rr(K)$ |
| 2 | $\epsilon$ | $\epsilon r(K)$ | $\cdots$ | $\epsilon r(K)$ |
| $\vdots$ | $\vdots$ | | | $\vdots$ |
| $K$ | $\epsilon$ | $\epsilon r(K)$ | $\cdots$ | $\epsilon r(K)$ |

When $\epsilon$ goes to zero, $\delta$ goes to zero and the efficiency ratio equals $r(K)$.

*For the case of $K > M$:* The value of $\epsilon$ in the table is chosen as $\frac{\delta R}{\{(K-1)-[r(K)+\delta][(M-2)(K-M+1)r(K-M+1)]\}r(K)}$. With the condition $Rr(K) > \epsilon$, the social optimum is achieved when the first player selects resource 1, $M-2$ players select each of the remaining resources and the $K-M+1$ players all select the last resource. In that case, the total payoff would be $R+(M-2)\epsilon r(K)+(K-M+1)\epsilon r(K)r(K-M+1)$ at the socially optimal solution. On the other hand, we can identify one Nash equilibrium $\boldsymbol{\sigma}$ where all players select resource 1. The total payoff at $\boldsymbol{\sigma}$ is $Rr(K)+(K-1)\epsilon r(K)$. Therefore, we have

$$\begin{aligned}
\mathrm{ER}(\boldsymbol{\sigma}) &= \frac{Rr(K)+(K-1)\epsilon r(K)}{R+(M-2)\epsilon r(K)+(K-M+1)\epsilon r(K)r(K-M+1)} \\
&= r(K) + \frac{[(K-1)-r(K)(M-2)-r(K)(K-M+1)r(K-M+1)]\epsilon r(K)}{R+(M-2)\epsilon r(K)+(K-M+1)\epsilon r(K)r(K-M+1)} \\
&= r(K)+\delta,
\end{aligned}$$

Furthermore, we show that $\epsilon$ is an increasing function of $\delta$.

$$\begin{aligned}
\epsilon &= \frac{\delta R}{\{(K-1)-[r(K)+\delta][(M-2)(K-M+1)r(K-M+1)]\}r(K)} \\
\frac{\partial \epsilon}{\partial \delta} &= \frac{\{(K-1)-[r(K)+\delta][(M-2)(K-M+1)r(K-M+1)]\}r(K)R-\delta R\{-[(M-2)(K-M+1)r(}{\{(K-1)-[r(K)+\delta][(M-2)(K-M+1)r(K-M+1)]\}^2 r^2(K)} \\
&= \frac{\{(K-1)-r(K)[(M-2)(K-M+1)r(K-M+1)]\}r(K)R}{\{(K-1)-[r(K)+\delta][(M-2)(K-M+1)r(K-M+1)]\}^2 r^2(K)} > 0
\end{aligned}$$

When $\epsilon$ goes to zero, $\delta$ goes to zero and the efficiency ratio equals $r(K)$. ■

*Theorem 2:* For the family of asymmetric $(K,M)$-game with $M>1$, the worst-case PoA is $r(K)$.

*Proof:* Lemmas 3 and 4 together lead to Theorem 2. ■

## VI. PoA Analysis of Symmetric Games

We now consider the intermediate class of symmetric games where constraints are imposed on the preference constants across players and resources. The class of symmetric games can be further divided into two categories: player-specific symmetric games and resource-specific symmetric games. The worst-case PoA is more difficult to analyze in these games than the asymmetric games due to the additional constraints imposed on the preference constants. In some cases we are only able to provide exact worst-case PoA for specific choices of the allocation function $r(n)$.

### A. Player-specific Symmetric Games

For player-specific symmetric game, different players have different preferences for resources, but each player has the same preference constant for different resources, i.e., $R_m^k = R^k$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$.

Without loss of generality, we assume players are arranged in descending order of their preference constants, i.e., $R^1 \geq R^2 \geq ... \geq R^K$.

*Proposition 1:* For a player-specific symmetric $(K, M)$-game, the total payoff at a socially optimal solution is

$$opt = \begin{cases} \sum_{k=1}^{K} R^k, & \text{if } K \leq M. \\ \sum_{k=1}^{M-1} R^k + r(K-M+1) \sum_{k=M}^{K} R^k, & \text{if } K > M. \end{cases}$$

*Proof:* Using similar argument as in Lemma 2, we show that players will select different resources in the socially optimal solution for $K \leq M$. For the case of $K > M$, we first show that for any two resources that are shared among players, it is always optimal to allocate a resource to the largest player, and let the remaining players share the other resource. This can be shown by the fact that $R_i[r(1) - r(n_1)] \geq R_j[r(n_2) - r(n_1 + n_2 - 1)]$ for index $i < j$. We then check from the largest player to see if he owns a resource solely. We continue with the other players in an ascending order. As a result, the first $M-1$ players all selects a resource solely and the remaining players shared the last resource. ∎

*Theorem 3:* For the family of player-specific symmetric $(K, M)$-game, the worst-case PoA

$$\min_{\boldsymbol{R}} PoA(K, M, \boldsymbol{R}) = \begin{cases} 1, & \text{if } K \leq M. \\ r(\lfloor \frac{K}{M} \rfloor), & \text{if } K > M \text{ and } (K \mod M) = 0. \\ r(\lfloor \frac{K}{M} \rfloor + 1), & \text{if } K > M \text{ and } (K \mod M) \neq 0. \end{cases}$$

*Proof:* The best response of player $k \in \mathcal{K}$ at a Nash equilibrium is $\sigma_k$ if and only if $R^k_{r(n_{\sigma_k})} \geq R^k_j r(n_j+1), \ \ \forall j \in \mathcal{M}$ and $j \neq \sigma_k$. Since the preference constants of all resources for a single player is the same in a player-specific symmetric game, each player's best response is similar to that in identical games. This results in an "even" distribution of players on all resources, where the difference of number of players on any two resources is no greater than 1 in any Nash equilibrium, i.e., $|n_i - n_j| \leq 1, \ \ i,j \in \mathcal{M}$.

For $K \leq M$, the "even" distribution of players in Nash equilibrium implies that every player selects a different resource, where is the same as in the socially optimal solution. Therefore, PoA=1.

For $K > M$, the total payoff at both the Nash equilibrium $\boldsymbol{\sigma}$ and the social optimum can be determined. In particular, if $(K \mod M) = 0$, then the efficiency ratio is

$$ER(\boldsymbol{\sigma}) = \frac{r(\lfloor \frac{K}{M} \rfloor) \sum_{k=1}^{K} R^k}{\sum_{k=1}^{M-1} R^k + r(K-M+1) \sum_{k=M}^{K} R^k}.$$

This can be lower-bounded by $r(\lfloor \frac{K}{M} \rfloor)$ as $r(K-M+1) \leq 1$. This lower-bound can be achieved when some $K-M+1$ players have a preference constant that is significantly small when compared with the other $M-1$ players, based on a similar argument as Lemma 4.

For the case when $(K \mod M) \neq 0$, we can show that the efficiency ratio is lower-bounded by $r(\lfloor \frac{K}{M} \rfloor + 1)$ and is achievable. ■

For the family of player-specific symmetric games, we can identify the possible Nash equilibria, which achieve an even distribution of players on different resources. Hence, PoA is known once the preference constants $R^k_m$s are known. The worst-case PoA results from a significant difference in the preference constants of players.

### *B. Resource-specific Symmetric Games*

For resource-specific symmetric game, all players have the same preference constant for the same resource, but have different preferences for different resources, i.e., $R^k_m = R_m$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$.

Without loss of generality, we assume resources are arranged in descending order of preference constant of players, i.e., $R_1 \geq R_2 \geq ... \geq R_M$.

Consider $K$ players in a symmetric game. Denote $\sigma_k$ to be the resource selected by player $k$.

Then, the total payoff of the players is

$$\sum_{k\in\mathcal{K}} \pi^k_{\sigma_k} = \sum_{k\in\mathcal{K}} R_{\sigma_k} r(n_{\sigma_k}) = \sum_{m\in\mathcal{M}} R_m n_m r(n_m).$$

With the new sharing function defined under Assumption 1, the total payoff received by the $K$ players can be written as a sum of payoffs generated by each resource, i.e., $\sum_{m\in\mathcal{M}} R_m f(n_m)$ where $\sum_{m\in\mathcal{M}} n_m = K$.

*Proposition 2:* For a resource-specific symmetric $(K, M)$-game, the total payoff at a socially optimal solution is

$$opt = \begin{cases} \sum_{j=1}^{K} R_j, & \text{if } K \leq M. \\ \sum_{j=1}^{M-1} R_j + R_M f(K-M+1), & \text{if } K > M. \end{cases}$$

*Proof:* Consider the case when the number of players is no larger than the number of resources (i.e., $K \leq M$). We first show that if there exists a (congested) resource with more than 1 player and another resource with no player, we can always improve the total payoff by switching a player from the congested resource to the unused resource.

Consider a case with two resources, where one is selected by $K$ players and the other remains unused. Suppose $K$ players selecting a single resource is a socially optimal solution. Without loss of generality, we assume all players are selecting the first resource. In this case, the optimum would be $R_1 f(K)$. Now, we consider the scenario when one of the $K$ players switches from the congested resource to the unused resource. The new total payoff is $R_1 f(K-1) + R_2 f(1)$. Since $f(n)$ is decreasing under assumption 1, the new total payoff is greater than the original $R_1 f(K)$. It leads to a contradiction. Therefore, both resources are selected at the social optimum.

Using similar argument as in Lemma 2, we then generalize the proof to the $K$ player case that $K$ resources are selected at a socially optimal solution. To be specific, the optimal is the sum of the preference constants for the first $K$ resources.

Now let us consider the case where there are more players than resources (i.e., $K > M$). At the social optimum, the first $M-1$ resources are selected by one player each, and resource $M$ is selected by the remaining $K-M+1$ players. We provide a constructive proof by assigning players to resources one by one. We first assign $M$ players to $M$ resources, one on each resource. Then we can show that adding the an additional player to the smallest resource $M$ leads to the minimum performance degradation.

When all the $M$ resources are selected by exactly one player, the loss in total payoff generated by the $(M+1)$th player for selecting resource $j$ is given by $R_j[f(1)-f(2)]$. In order to minimize the loss, the $(M+1)$th player should select resource $M$ that has the smallest preference constant,$R_M$. Similarly, the loss generated by the $(M+2)th$ player is $R_M[f(2)-f(3)]$ or $R_j[f(1)-f(2)]$ if $j \neq M$. Since $f(n)$ is convex and non-increasing, $f(2)-f(3)$ is smaller than $f(1)-f(2)$ and hence $R_M[f(2)-f(3)]$ is smaller than $R_j[f(1)-f(2)]$. To minimize the additional loss, we also assign the $(M+2)$th player to resource $M$. Similarly, we continue to add the rest of the $K-M$ players, and show that all these players should be added to the smallest resource $M$ to minimize the social welfare loss. As a result, the first $M-1$ resources are selected by one player each, and resource $M$ is selected by the remaining $K-M+1$ players. Hence, the social optimum is $\sum_{j=1}^{M-1} R_j + R_M f(K-M+1)$. ■

Since the computation of worst-case PoA for this family of games is rather complicated, we only consider a special allocation function of $r(n)=\frac{1}{n}$. It captures a special feature that $f(n)=nr(n)$ is always equal to 1. This implies that the total payoff of a resource is independent of the number of players as long as it is being selected.

*Theorem 4:* For the family of resource-specific symmetric $(K,M)$-game with more players than number of resources (i.e., $K \geq M$) and $r(n)=\frac{1}{n}$, the worst-case PoA is $\frac{K}{K+M-1}$.

*Proof:* We prove the theorem in two steps.

*Step I (lower bound): we first prove that the efficiency ratio of a Nash equilibrium is lower bounded by* $\frac{K}{K+M-1}$. We assume that there exists a Nash equilibrium $\boldsymbol{\sigma}$ with a set of $h$ resources not selected by any players, denoted by $\mathcal{H}=\{M-h+1,...,M\}$ where $0 \leq h \leq M-1$. This implies that each of the $K$ players selects one of the first $M-h$ resources, i.e., $\sum_{m\in\mathcal{M}\setminus\mathcal{H}} n_m = K$. With the best-reply strategy at Nash equilibrium, i.e., $\frac{R_j}{n_j} \geq R_k$ for all $j \in \mathcal{M}\setminus\mathcal{H}$ and $k \in \mathcal{H}$, we have $\frac{R_j}{n_j} \geq \frac{1}{h}\sum_{k\in\mathcal{H}} R_k$ for all $j \in \mathcal{M}\setminus\mathcal{H}$.

Now we can derive the *lower bound* of the efficiency ratio

$$
\begin{aligned}
ER(\boldsymbol{\sigma}) = \frac{SUM(\boldsymbol{\sigma})}{opt} &= \frac{\sum_{j\in\mathcal{M}} R_j I_j}{\sum_{j\in\mathcal{M}} R_j} = \frac{\sum_{j\in\mathcal{M}\setminus\mathcal{H}} R_j}{\sum_{j\in\mathcal{M}} R_j} = 1 - \frac{\sum_{j\in\mathcal{H}} R_j}{\sum_{j\in\mathcal{M}\setminus\mathcal{H}} R_j + \sum_{j\in\mathcal{H}} R_j} \\
&\geq 1 - \frac{\sum_{j\in\mathcal{H}} R_j}{\sum_{j\in\mathcal{M}\setminus\mathcal{H}}(\frac{n_j}{h}\sum_{k\in\mathcal{H}} R_k) + \sum_{j\in\mathcal{H}} R_j} = 1 - \frac{\sum_{j\in\mathcal{H}} R_j}{\frac{1}{h}\sum_{k\in\mathcal{H}} R_k(\sum_{j\in\mathcal{M}\setminus\mathcal{H}} n_j) + \sum_{j\in\mathcal{H}} R_j} \\
&= 1 - \frac{\sum_{j\in\mathcal{H}} R_j}{\frac{N}{h}\sum_{k\in\mathcal{H}} R_k + \sum_{j\in\mathcal{H}} R_j} = 1 - \frac{1}{\frac{N}{h}+1} = \frac{N}{N+h}.
\end{aligned}
$$

The above analysis is for a fixed $h$. It is clear that the lower bound $\frac{N}{N+h}$ is a decreasing function of $h$, and achieves its minimum value of $\frac{N}{N+M-1}$ with $h = M-1$ resources not being selected at a Nash equilibrium $\boldsymbol{\sigma}$.

*Step II (achievability): we construct a resource-specific symmetric $(K, M)$-game that achieves the efficiency ratio $\frac{K}{K+M-1}$.* Consider a game with $M-1$ identical resources having same preference constant $R_j = R$, while the preference constant for the remaining resource is $KR$. A Nash equilibrium $\boldsymbol{\sigma}$ of this game is that all players select the resource with the largest preference constant. The efficient ratio of $\boldsymbol{\sigma}$ is

$$ER(\boldsymbol{\sigma}) = \frac{SUM(\boldsymbol{\sigma})}{opt} = \frac{KR}{KR + (M-1)R} = \frac{K}{K+M-1}.$$

Combining both steps and the definition of PoA, we have proved the theorem. ∎

*Theorem 5:* For the family of resource-specific symmetric $(K, M)$-game with less users than number of channels (i.e., $K < M$) and $r(n) = \frac{1}{n}$, the worst-case PoA is $\frac{K}{2K-1}$.

*Proof:* First arrange the resources in the descending order of preference constants. Since the last $M - N$ resources will not be selected either in the socially optimal solutions or any Nash equilibrium, we can safely discard them. Therefore, the problem is reduced to a resource-specific symmetric $(K, K)$-game and Theorem 4 applies. ∎

For general non-increasing $r(n)$ functions, the exact worst-case PoA is hard to compute. The difficulty mainly lies in the fact that congestion vectors in any Nash equilibrium are coupled with different preference constants and hence are hard to eliminate. We try to give the lower-bound of all Nash equilibria in Appendix A.

## VII. A Wireless Application

An application of singleton congestion games with preference constants is spectrum sharing in a cognitive radio wireless network. Consider $M$ channels owned by $M$ primary licensed users (PUs), and $K$ secondary unlicensed users (SUs) who want to share the channels whenever the channels are not used by the PUs. The time is divided into discrete slots. The PU of a particular channel may transmit or be silent in any given time slot. A SU needs to decide which channel to sense at the beginning of the time slot. If the channel turns out to be occupied by the PU, the SU will remain silent for the rest of the time slot. If the channel is idle, then the SU will try to access the channel. If there are more than one SU sensing the same idle channel, then the successful transmission probability depends on the choice of medium access control (MAC) schemes (as explained in detail shortly).

From a SU $k$'s point of view, the maximum expected data rate received for being the sole user sensing channel $m$ is $R_m^k$. This takes into account of the statistics of the PU's activity on this channel.[6] For example, if a SU can achieve a data rate of 10Mbps (if transmitting alone) on a particular channel when it is idle, and the channel is idle $30\%$ of the time, then the maximum expected data rate is 3Mbps. Due to different transmission technologies and locations, different SUs may achieve different data rates on the same channel. Due to hardware limitation, we assume that each SU can only sense one channel in a time slot and transmit on the same channel if it is idle. A SU's goal is to select a channel to sense in order to maximize his expected data rate (by considering both the channel availability and the congestion effect). Such optimization not only depends on $R_m^k$s but also on the number of SUs competing for the same channel.

Different values of preference constants $R_m^k$ lead to different families of congestion games. Identical games represent the scenario where all channel conditions and hardware capabilities of SUs are the same. With a slight variation, player-specific symmetric games refer to systems where SUs have different choices of transmission technologies though the channels are all identical. In contrast, resource-specific symmetric games refer to systems where there exists channels with different bandwidths though SUs are identical. Lastly, SUs in different locations usually experience different channel conditions. This most general scenario is modeled as asymmetric games.

We assume that each SU sensing channel $m$ has an equal probability $r(n_m)$ of succeeding in his transmission, where $n_m$ is the number of SUs sensing channel $m$. We assume $r(n_m)$ has the following properties:

- SU $k$ achieves an expected data rate of $R_m^k$ when it is the only user sensing channel $m$. Therefore, $r(1) = 1$ by definition.
- The more SUs on a single channel, the less chance each SU transmits successfully, i.e., $r(n_m)$ is a decreasing function of $n_m$.
- The sum of success probabilities of all SUs cannot exceed 1, i.e., $n_m r(n_m) \leq 1$.

[6] For simplicity, we assume that the SUs do not take historical sensing result into consideration.

TABLE IV: Price of anarchy for various cognitive uniform MAC games

| | Identical | Player-specific Symmetric | Resource-specific symmetric | Asymmetric |
|---|---|---|---|---|
| $\min_{\boldsymbol{R}} PoA(K, M, \boldsymbol{R})$ | 1 | $\frac{1}{\lfloor \frac{K}{M} \rfloor}$ [7] or $\frac{1}{(\lfloor \frac{K}{M} \rfloor + 1)}$ | $\frac{K}{K+\min(K,M)-1}$ | $\frac{1}{K}$ |
| $\min_{K,M,\boldsymbol{R}} PoA(K, M, \boldsymbol{R})$ | 1 | 0 | $\frac{1}{2}$ | 0 |

The specific form of $r(\cdot)$ depends on the MAC scheme. In the following, we will consider two cases: uniform MAC and slotted Aloha.

### *A. Uniform MAC*

When there are $n$ SUs competing for a channel, the simplest way to resolve the conflict is to allow each SU to grab the channel with an equal probability $\frac{1}{n}$. This can be achieved as follows. A SU who has sensed an idle channel will pick a random countdown value within a fixed time window $Y$ and continue to sense the channel for presence of other SUs. A SU will proceed to transmit if his countdown timer expires and no other SUs have started the transmission on the channel. Otherwise, if the channel is being used, the SU loses the opportunity to sense or transmit in other channels and remains idle till the next slot. In this case, only one of the many SUs can transmit on a channel at a particular time slot. Assuming $Y$ is large enough, then the probability of getting the channel is $\frac{1}{n}$. This simple model captures the case where competition only introduces uncertainty in terms of who can access the channel without wasting the resource. The slotted Aloha model discussed next examines resource waste due to competition.

Under this uniform MAC scheme, we can compute the exact worst-case PoA for all families of games as shown in Table IV, with numerical results in Figure 1, where there are $M = 10$ channels and the number of SUs $K$ varies from 1 to 80.

In uniform MAC, channels are shared efficiently without loss. In identical games, SUs are evenly distributed on all channels at any Nash equilibrium. Thus the PoA is 1 in the identical games independent of $K$ as shown in Figure 1.

An interesting observation in resource-specific symmetric games is that the PoA first decreases, then increases, and finally converges to 1 (when $K$ is large enough; not shown in the figure). The

[7] The PoA equals to $\frac{1}{\lfloor \frac{K}{M} \rfloor}$ if $(K \mod M) = 0$ and equals to $\frac{1}{(\lfloor \frac{K}{M} \rfloor + 1)}$ otherwise.

drop at the beginning is mainly due to the incomplete usage of channels when the number of SUs is small. It is because different channels can provide different transmission rates. The worst-case PoA happens when there exists a channel that is significantly better than the others, so that SUs tend to sense the same channel and leave other channels unused. With an increasing number of SUs, the probability of having unused channels reduces. If the number of SUs increases tremendously, all channels are selected eventually. Hence, the worst-case PoA approaches 1. The universal worst-case PoA happens when number of channels and SUs are the same (e.g., $K = M = 10$ in Figure 1), which can be as bad as $\frac{1}{2}$. This is $50\%$ worse than the one obtained in identical games.

Similar to the resource-specific symmetric games, the worst-case PoA for asymmetric games happens when all SUs selects the same channel while ignoring all other channels. One notable difference is that the PoA is independent of the number of channels $M$ (not reflected in the figure as we only have $M = 10$ here). Since $r(K)$ is a decreasing function, the PoA is smaller for larger number of SUs. The PoA can go to zero when $K$ is large enough. From a system design point of view, in order to minimize the performance loss due to the lack of coordination among SUs, it is a good idea to limit the number of competing SUs in a single network. In other words, it may make sense to organize channels into smaller, separate systems, thereby limiting the number of competing users. Consider a scenario with $M = 20$ and $K = 100$, in which case $PoA = r(100)$ (i.e., $PoA = 0.01$ for the case of $f(n) = 1$). If we divide the system into 4 sub-systems with $M = 5$ and $K = 25$, then $PoA = r(25)$ (i.e., $PoA = 0.04$ for the case of $f(n) = 1$).

The family of player-specific symmetric games is an intermediate between identical games and asymmetric games. With the same preference constants for each SU on each channel, we can identify the Nash equilibria to be an even distribution of SUs of all channels. When we have more SUs (a larger $K$), it is more likely that the variations among SUs becomes larger, and thus the worst-case PoA gets worse.

### *B. Slotted Aloha*

Here we look at another common MAC protocol, the slotted Aloha, where the competition among SUs leads to resource waste.

After a SU senses an idle channel, it will decide whether to contend for the channel (immediately). A SU can successfully get the channel when he is the only one transmits. If two or more SUs transmit in the same channel, all transmissions fail.

To ensure fairness among SUs, we assume all SUs sensing the same idle channel have the same transmission probability independent of the data rates the SUs receive. Given the number of SUs on the same channel $n$, the probability for a SU to transmit successfully is given by $r(p,n) = p(1-p)^{n-1}$. Since each SU wants to maximize his expected payoff, it is equivalent to selecting the common transmission probability $p$ to maximize $r(p,n)$. We can show that the under the optimal choice of $p$, we have

$$r(n) = (\frac{1}{n})(1-\frac{1}{n})^{n-1} \text{ and } f(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ (1-\frac{1}{n})^{n-1}, & \text{if } n > 1 \end{cases}.$$

We verify that the two functions above satisfy Assumption 1. The computation can be found in Appendix B. The worst-case PoA can be computed by plugging in the function $r(n)$ and $f(n)$ into the results in Sections IV, V, and VI.

We illustrate the worst-case PoA for the slotted Aloha scheme in Figure 2. Here the number of channels $M = 10$ and the number of SUs $K$ varies between 1 and 60.

When the number of SUs is smaller than channels (i.e., $K < M$) and channels are identical to users (identical games or player-symmetric games), there is no congestion and $PoA = 1$. When the number of SUs is larger than the number of channels, congestion happens and some resources are wasted. The inefficiency of Nash equilibria due to selfish behaviors becomes more severe as the number of SUs $K$ increases. In identical games, SUs tend to spread out in a Nash equilibrium, which leads to significant losses comparing to the social optimum where the loss is restricted to a single channel only. The situation is worse in asymmetric games where channels are different. It is possible for all SUs to sense the same channel and this congestion of users induces loss to the total payoff and hence a worse performance in the Nash equilibrium. The worst-case PoA for identical games converges to $\lim_{K\to\infty} f(K) = \frac{1}{e}$ when the number of SUs increases. While for that of player-specific symmetric and asymmetric games, the worst-case PoA tends to $\lim_{K\to\infty} r(K) = 0$.

When comparing uniform MAC and slotted Aloha, the worst-case PoA obtained in slotted Aloha is in general smaller than that in uniform MAC in different families of games. We also note that the worst-case PoA is getting smaller when we allow a higher degree of freedom in the choices of preference constants. Hence, it is decreasing in the order of identical games, symmetric games, and asymmetric games.

## VIII. Conclusion

In this paper we derive the exact worst-case PoA for singleton congestion games with different preference constants. We show that the worst-case PoA decreases (or gets worse) in the order of identical games, symmetric games, and asymmetric games. We also identify several possible outcomes that lead to worst-case PoA. Using these results we can design systems with smaller efficiency loss by controlling the number of players competing for resources, or controlling the heterogeneity among different resources and players. In addition, we apply the general model of singleton congestion games to wireless cognitive radio networks with two medium access control schemes: uniform MAC and slotted Aloha. The worst-case PoA obtained in slotted Aloha is in general smaller than that in uniform MAC in all four families of games, due to the resource waste in the slotted Aloha.

## Appendix

### A. Supplement for lower-bound of Nash equilibria in resource-specific symmetric games

We first start with illustrating some facts are later used in the proof.

*Fact 1:* Consider a function $g(n)$ with properties $g'(n) > 0$ and $g''(n) \geq 0$. We have the following:

- Given an index set $\mathcal{M} = \{1, ..., |M|\}$. If $K > |\mathcal{M}|$, $n_1 = K - |\mathcal{M}| + 1$ and $n_j = 1$ for all $j \in \mathcal{M} \neq 1$ is an optimal solution to the optimization problem,

$$\begin{aligned} \underset{(n_i)_{i\in\mathcal{M}},\{n_k\}_{k\in\mathcal{M}\neq i}}{\text{Maximize}} \quad & \sum_{k\in\mathcal{M}\setminus\{i\}} \frac{1}{r(n_k+1)} + \frac{1}{r(n_i)} \\ \text{subject to} \quad & n_1 \geq n_k \geq 1, \;\; \forall k \in \mathcal{M} \\ & \sum_{k\in\mathcal{M}} n_k = K \\ & n_k \in \mathbb{N}^+, \;\; \forall k \in \mathcal{M}. \end{aligned} \tag{2}$$

- For any $y > x$,

$$g(y) + (y-x)g(1) > g(x) + (y-x)g(2). \tag{3}$$

We assume that there exists a Nash equilibrium $\boldsymbol{\sigma}$ with a set of $h$ resources not selected by any players, denoted by $\mathcal{H} = \{M-h+1, ..., M\}$ where $0 \leq h \leq M-1$.

From Proposition 2, we obtain the social optimum in resource-specific symmetric games, $opt = \sum_{j=1}^{M-1} R_j + R_M f(K-M+1)$. To simplify the expression, we introduce a new variable $I_j$ as follows,

$$I_j = \begin{cases} 1, & \text{if } j \neq M. \\ f(K-M+1), & \text{if } j = M. \end{cases}$$

We can re-write the expression of social optimum as $opt = \sum_{m \in \mathcal{M}} R_m I_m$.

The efficiency ratio is

$$ER(\boldsymbol{\sigma}) = \frac{\sum_{m \in \mathcal{M} \setminus \mathcal{H}} R_m f(n_m)}{\sum_{m \in \mathcal{M}} R_m I_m} = \frac{\sum_{m \in \mathcal{M} \setminus \mathcal{H}} R_m n_m r(n_m)}{\sum_{m \in \mathcal{M}} R_m I_m}.$$

We first lower-bound the ratio by considering the best response of players at Nash equilibrium. We then show that the bound is smallest when $h$ takes the value of $M-1$. Hence, we obtain a non-trivial lower-bound of the efficiency ratio, $\frac{f(K)}{1+r(K)(M-2)+r(K)f(K-M+1)}$.

Denote $\hat{\jmath} = \arg\min_{j \in \mathcal{M} \setminus \mathcal{H}} R_j r(n_j)$. Together with the fact that each of the $K$ players selects one of the first $M-h$ resources, i.e., $\sum_{m \in \mathcal{M} \setminus \mathcal{H}} n_m = K$, we have

$$ER(\boldsymbol{\sigma}) \geq \frac{\sum_{m \in \mathcal{M} \setminus \mathcal{H}} n_m R_{\hat{\jmath}} r(n_{\hat{\jmath}})}{\sum_{m \in \mathcal{M}} R_m I_m} = \frac{K R_{\hat{\jmath}} r(n_{\hat{\jmath}})}{\sum_{m \in \mathcal{M}} R_m I_m}.$$

We can further lower-bound the efficiency ratio by considering the best response of player $\hat{\jmath}$ at Nash equilibrium: $R_{\hat{\jmath}} r(n_{\hat{\jmath}}) \geq R_j$ for all $j \in \mathcal{H}$ and $R_{\hat{\jmath}} r(n_{\hat{\jmath}}) \geq R_i r(n_i+1)$ for all $i \in \mathcal{M} \setminus \mathcal{H}$ and $i \neq \hat{\jmath}$.

$$\begin{aligned} ER(\boldsymbol{\sigma}) &= \frac{K R_{\hat{\jmath}} r(n_{\hat{\jmath}})}{\sum_{i \in \mathcal{M} \setminus \mathcal{H}} R_i I_i + \sum_{j \in \mathcal{H}} R_j I_j} \\ &\geq \frac{K R_{\hat{\jmath}} r(n_{\hat{\jmath}})}{\sum_{i \in \mathcal{M} \setminus \mathcal{H} \setminus \{\hat{\jmath}\}} \frac{r(n_{\hat{\jmath}}) R_{\hat{\jmath}}}{r(n_i+1)} I_i + R_{\hat{\jmath}} I_{\hat{\jmath}} + \sum_{j \in \mathcal{H}} R_{\hat{\jmath}} r(n_{\hat{\jmath}}) I_j} \\ &= \frac{K}{\sum_{i \in \mathcal{M} \setminus \mathcal{H} \setminus \{\hat{\jmath}\}} \frac{I_i}{r(n_i+1)} + \frac{I_{\hat{\jmath}}}{r(n_{\hat{\jmath}})} + \sum_{j \in \mathcal{H}} I_j}. \end{aligned}$$

We now consider the case when resource $M$ is not selected by any players, i.e., $M \in \mathcal{H}$. This implies that $I_i = 1$ for $i \in \mathcal{M} \setminus \mathcal{H}$. The problem is reduced to

$$ER(\boldsymbol{\sigma}) = \frac{K}{\sum_{i \in \mathcal{M} \setminus \mathcal{H} \setminus \{\hat{\jmath}\}} \frac{1}{r(n_i+1)} + \frac{1}{r(n_{\hat{\jmath}})} + [(h-1) + f(K-M+1)]}.$$

To further lower-bound the efficiency ratio, we transform the first two terms in the denominator into a maximization problem as shown in (2). With the condition that $\frac{1}{r(n)}$ is concave increasing, it satisfies the conditions on $g(n)$ in (2). Since $i$ is in the set of chosen resources, the number of players choosing resource $i$, $n_i$ should be at least 1. In addition, we have the condition that $n_1$ is the largest and the sum of players is $K$, i.e., $\sum_{i\in\mathcal{M}\setminus\mathcal{H}} n_i = K$. Thus, the solution of this maximization problem is given by $n_1 = K - M + h + 1$ and $n_j = 1$ for all $j \in \mathcal{M} \neq 1$.

For the case of $\hat{\jmath} = 1$:

$$\begin{aligned} ER(\boldsymbol{\sigma}) &\geq \frac{K}{\frac{1}{r(K-M+h+1)} + \sum_{m\in\mathcal{M}\setminus\mathcal{H}\setminus\{1\}} \frac{1}{r(1+1)} + (h-1) + f(K-M+1)} \\ &= \frac{K}{\frac{1}{r(K-M+h+1)} + (M-h-1)\frac{1}{r(2)} + (h-1) + f(K-M+1)} \\ &> \frac{K}{\frac{1}{r(K)} + (M-h-1)\frac{1}{r(1)} + (h-1) + f(K-M+1)} \\ &= \frac{K}{\frac{1}{r(K)} + (M-2) + f(K-M+1)}. \end{aligned}$$

The last inequality is derived from (3). Similar argument for the case of $\hat{\jmath} \neq 1$.

Finally, we consider the remaining case when resource $M$ is chosen. The efficiency ratio at Nash equilibrium is again lower-bounded by $\frac{K}{\frac{1}{r(K)}+(M-2)+f(K-M+1)}$ with

$$\begin{aligned} ER(\boldsymbol{\sigma}) &= \frac{\sum_{m\in\mathcal{M}} R_m f(n_m)}{\sum_{m\in\mathcal{M}\setminus\{M\}} R_m + R_M f(K-M+1)} \\ &> \frac{\sum_{m\in\mathcal{M}} R_m f(K)}{\sum_{m\in\mathcal{M}\setminus\{M\}} R_m + R_M f(K-M+1)} \\ &> f(K) \\ &> \frac{K}{\frac{1}{r(K)} + (M-2) + f(K-M+1)}. \end{aligned}$$

We conclude that the efficiency ratio is lower-bounded by $\frac{K}{\frac{1}{r(K)}+(M-2)+f(K-M+1)}$ in all possible Nash equilibria.

### *B. Verification of function in slotted Aloha*

Given the number of SUs on the same channel $n$, the probability for a SU to transmit successfully is given by $r(p,n) = p(1-p)^{n-1}$. Since each SU wants to maximize his expected payoff, it is equivalent to selecting the common transmission probability $p$ to maximize $r(p, n)$. By first order differentiation with respect to $p$,

$$\frac{\partial r(p,n)}{\partial p} = 0$$

$$(1-p)^{n-1} - p(n-1)(1-p)^{n-2} = 0$$

$$1 - p = p(n-1)$$

$$p^* = \frac{1}{n}$$

We can show that the under the optimal choice of $p = \frac{1}{n}$, we have

$$r(n) = (\frac{1}{n})(1-\frac{1}{n})^{n-1} \text{ and } f(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ (1-\frac{1}{n})^{n-1}, & \text{if } n > 1 \end{cases}.$$

We then compute the first and second derivatives of $f(n) = (1-\frac{1}{n})^{n-1}$.

*1) First Derivative:*

$$\begin{aligned} f(n) &= (1-\frac{1}{n})^{n-1} \\ \ln f(n) &= (n-1)\ln(1-\frac{1}{n}) \\ \frac{f'(n)}{f(n)} &= (n-1) \times \frac{1}{1-\frac{1}{n}} \times \frac{1}{n^2} + \ln(1-\frac{1}{n}) \\ f'(n) &= f(n)[\frac{1}{n} + \ln(1-\frac{1}{n})] \\ &< f(n)[\frac{1}{n} - \frac{1}{n}] \\ &= 0 \end{aligned}$$

The last inequality is due to the fact that $\ln(1+x) < x$.

*2) Second Derivative:*

$$
\begin{aligned}
f'(n) &= f(n)[\frac{1}{n} + \ln(1-\frac{1}{n})] \\
f''(n) &= f(n)[-\frac{1}{n^2} + (\frac{1}{1-\frac{1}{n}})(\frac{1}{n^2})] + f'(n)[\frac{1}{n} + \ln(1-\frac{1}{n})] \\
&= f(n)(\frac{1}{n^2})(\frac{n}{n-1} - 1) + f(n)[\frac{1}{n} + \ln(1-\frac{1}{n})]^2 \\
&= f(n)(\frac{1}{n^2(n-1)}) + f(n)[\frac{1}{n} + \ln(1-\frac{1}{n})]^2 \\
&> 0
\end{aligned}
$$

From the above computation, we have $f'(n) < 0$ and $f''(n) > 0$. We verify that the function satisfies Assumption 1.

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TABLE V: Key Notations in this paper

| Notation | Physical Meaning |
|---|---|
| $M$ | the number of resources |
| $\mathcal{M}$ | the set of common resources |
| $K$ | the number of players |
| $\mathcal{K}$ | the set of players |
| $\Sigma_k$ | the set of pure strategies for player $k$ |
| $\sigma_k$ | strategy selected by player $k$ |
| $\boldsymbol{\sigma}$ | strategy profile of all players |
| $\Pi$ | the set of strategy profiles from all players |
| $n_m$ | the number of players selecting resource $m$ |
| $\boldsymbol{n}(\boldsymbol{\sigma})$ | the congestion vector corresponding to $\boldsymbol{\sigma}$ |
| $R_m^k$ | the preference constant for player $k$ to select resource $m$ |
| $\pi_m^k$ | payoff of player $k$ for selecting resource $m$ |
| $SUM(\boldsymbol{\sigma})$ | total payoffs received by all players at a Nash equilibrium $\boldsymbol{\sigma}$ |
| $opt$ | social optimum: maximum total payoffs received by all players |
| $\mathcal{H}$ | the set of unused resources |
| $r(n_m)$ | the share each of $n_m$ players has for selecting resource $m$ |
| $f(n)$ | the total share of $n$ players selecting the same resource |


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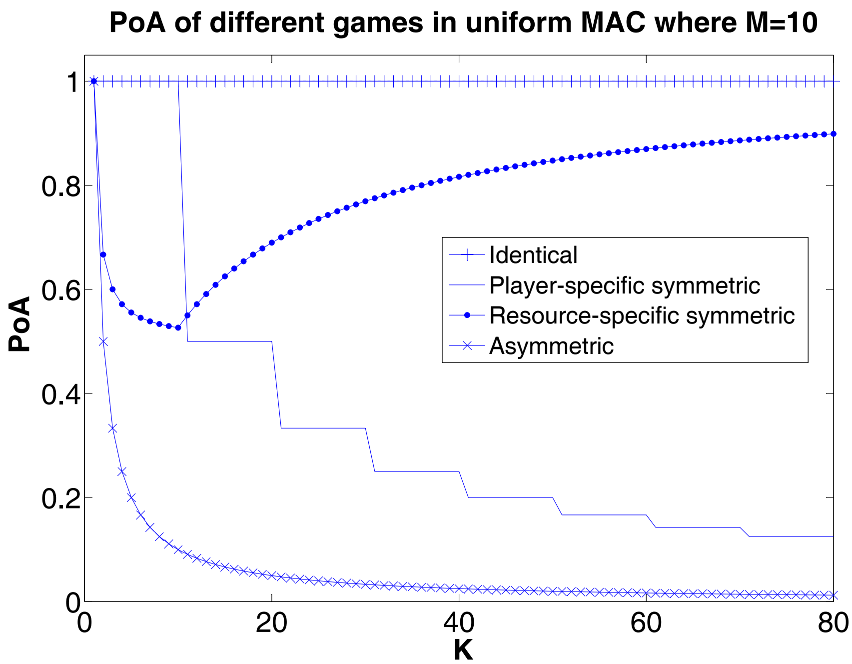


Fig. 1: PoA of different games in uniform MAC

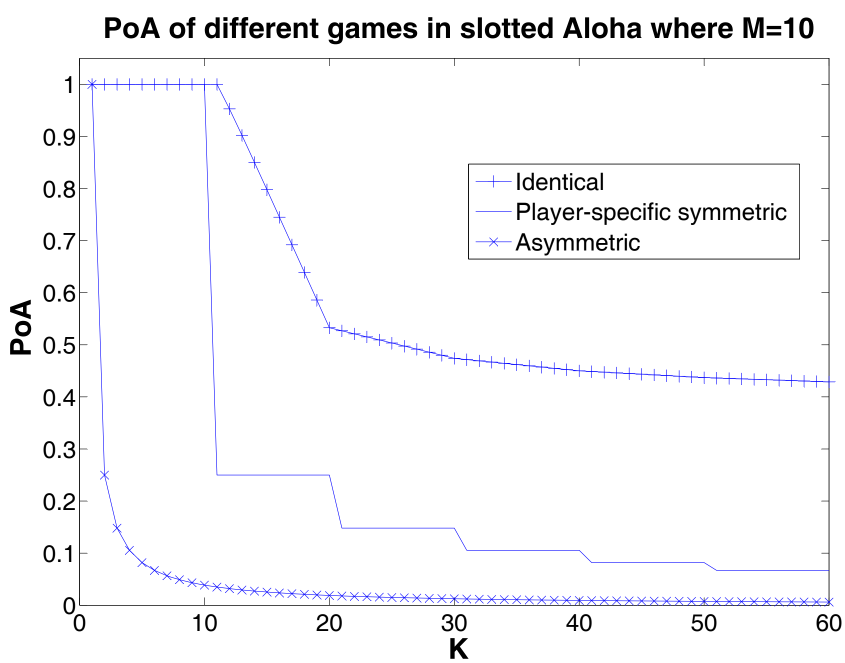


Fig. 2: PoA of different games in slotted Aloha