The Undecidability of Probabilistic Conditional Independence Implication

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Random Variables

- A **random variable (RV)** $X : \Omega \to \mathcal{X}$ is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a measurable space
- We focus on **discrete** random variables, i.e., the support $\mathcal{X}$ is finite or countable
  - Suffices to consider $\Omega = [0, 1]$ to be the standard probability space, i.e., $[0, 1]$ with the Lebesgue measure as the probability
- X, Y are (unconditionally) independent, denoted as $X \perp\!\!\!\perp Y$, if for all x, y,

$$\mathbb{P}((X, Y) = (x, y)) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

- X, Y are conditionally independent given Z, denoted as $X \perp\!\!\!\perp Y|Z$, if for all x, y, z,

$$\mathbb{P}((X,Y,Z) = (x,y,z))\mathbb{P}(Z = z) = \mathbb{P}((X,Z) = (x,z))\mathbb{P}((Y,Z) = (y,z))$$

- WLOG assume all random variables are positive-integer-valued, i.e., measurable functions $X : [0, 1] \to \mathbb{N}$

First-order Theory of Random Variables

- Consider first-order formulae (with logical symbols $\forall$, $\exists$, $\wedge$, $\vee$, $\neg$), with non-logical symbols $\cdot \perp\!\!\!\perp \cdot$ (unconditional independence) and $\cdot \perp\!\!\!\perp \cdot | \cdot$ (conditional independence)
- Variables in the formulae are random variables, i.e., measurable functions $[0,1] \to \mathbb{N}$
  - Just the ordinary first-order logic over the domain of measurable functions $[0,1] \to \mathbb{N}$, with the usual semantics
- Relation with probabilistic team semantics [Durand et al., 2018, Hannula et al., 2023]:
  - A probabilistic team $\mathbb{X}$ can be regarded as a joint distribution of the variables
  - Conditional independence $\mathfrak{A} \models_{\mathbb{X}} x \perp\!\!\!\perp_z y$ means $X \perp\!\!\!\perp Y | Z$ as RVs
  - Different semantics for $\vee$ and $\forall$

Undecidable Problems

- Undecidable problems are decision problems that cannot be solved by any algorithm
  - E.g., Halting problem [Turing, 1936], Diophantine equations [Matiyasevich, 1993], Wang tiles [Berger, 1966], word problem of groups [Novikov, 1955]
- We discuss the undecidability of:
  - Conditional independence implication problem
  - First-order theory of random variables with probabilistic independence relation
  - Conditional information inequalities
  - Network coding

Probabilistic Independence Implication Problem

- Determine whether a probabilistic independence relation among several random variables follows from a list of other such relations [Geiger et al., 1991, Matúš, 1994]
- E.g. $X \perp\!\!\!\perp Y \wedge XY \perp\!\!\!\perp Z \Rightarrow X \perp\!\!\!\perp YZ$
  - i.e., $\forall X, Y, Z. ((X \perp\!\!\!\perp Y \wedge XY \perp\!\!\!\perp Z) \rightarrow X \perp\!\!\!\perp YZ)$
  - In the language of probabilistic team semantics: $\mathfrak{A} \models_{\mathbb{X}} (x \perp\!\!\!\perp y \wedge xy \perp\!\!\!\perp z) \Rightarrow \mathfrak{A} \models_{\mathbb{X}} x \perp\!\!\!\perp yz$
- Geiger et al. [1991] gave a complete set of axioms:
  - (Triviality) $X \perp\!\!\!\perp \emptyset$
  - (Symmetry) $X \perp\!\!\!\perp Y \Rightarrow Y \perp\!\!\!\perp X$
  - (Decomposition) $X \perp\!\!\!\perp YZ \Rightarrow X \perp\!\!\!\perp Y$
  - (Mixing) $X \perp\!\!\!\perp Y \wedge XY \perp\!\!\!\perp Z \Rightarrow X \perp\!\!\!\perp YZ$
- Complete – all true probabilistic independence implications can be deduced from these axioms
- Hence probabilistic independence implication is **decidable**

Conditional Independence Implication Problem

- Determine whether a conditional independence relation among several random variables follows from a list of other such relations [Dawid, 1979, Spohn, 1980, Mouchart and Rolin, 1984]
- E.g. $X \perp\!\!\!\perp Y|Z \wedge X \perp\!\!\!\perp W|YZ \Rightarrow X \perp\!\!\!\perp W|Z$
- **Decidable** if all random variables have bounded cardinalities [Geiger and Meek, 1999, Niepert, 2012]
  - Follows from the decidability of the real polynomial equations
  - Hannula et al. [2019] – in EXPSPACE if all RVs are binary
- What about the case where the cardinalities of the random variables are not bounded?

Semi-graphoid Axioms

- Pearl and Paz [1987] proposed the following 4 axioms:
  - (Symmetry) $X \perp\!\!\!\perp Y|Z \;\Rightarrow\; Y \perp\!\!\!\perp X|Z$
  - (Decomposition) $X \perp\!\!\!\perp YW|Z \;\Rightarrow\; X \perp\!\!\!\perp Y|Z$
  - (Weak union) $X \perp\!\!\!\perp YW|Z \;\Rightarrow\; X \perp\!\!\!\perp Y|ZW$
  - (Contraction) $X \perp\!\!\!\perp Y|Z \wedge X \perp\!\!\!\perp W|YZ \;\Rightarrow\; X \perp\!\!\!\perp YW|Z$
- CI implication would be decidable if semi-graphoid axioms are complete (i.e., all true CI implications can be deduced from these axioms)
  - Simply apply the axioms repeatedly on every combination of random variables until we obtain the desired CI statement
- For the special case where every CI statement involves all random variables (saturated CI), semi-graphoid axioms are complete, and hence **decidable** [Malvestuto, 1992, Geiger and Pearl, 1993]
- Unfortunately, semi-graphoid axioms are incomplete [Studený, 1989]
- Is conditional independence implication decidable in general?

Undecidability of Conditional Independence Implication

- Studený [1989]: Semi-graphoid axioms [Pearl and Paz, 1987] are incomplete
  - Is it possible to add more axioms to make it complete?
- Studený [1992]: No, conditional independence has no finite axiomization
  - Does not rule out other kinds of algorithms
- Herrmann [1995]: Embedded multivalued database dependency is undecidable
- Li [2021]: CI implication is undecidable if one of the RVs is binary
- Li [2022a]: First-order theory of random variables with probabilistic independence relation is undecidable
  - Allow any combination of $\perp\!\!\!\perp, \forall, \exists, \wedge, \vee, \neg$, not only implication
- Li [2022b]: CI implication is **undecidable**
  - Uses the ideas of Herrmann [1995]
- Kühne and Yashfe [2022]: Another concurrent proof of undecidability via matroid theory

First-order Theory of Probabilistic Independence

- Consider first-order formulae with only one non-logical symbol $\perp\!\!\!\perp$ (probabilistic independence)
  - Variables are random variables $(X, Y, \ldots)$
- How to define condition that X is constant, written as $X \overset{\iota}{=} \emptyset$?
  - $X \overset{\iota}{=} \emptyset \;\Leftrightarrow\; X \perp\!\!\!\perp X$
- How to define relation that X is a function of Y, written as $X \overset{\iota}{\leq} Y$?
  - $X \overset{\iota}{\leq} Y \;\Leftrightarrow\; \forall U. (U \perp\!\!\!\perp Y \to U \perp\!\!\!\perp X)$
  - Write $X \overset{\iota}{=} Y \;\Leftrightarrow\; X \overset{\iota}{\leq} Y \wedge Y \overset{\iota}{\leq} X$ and
    $X \overset{\iota}{<} Y \;\Leftrightarrow\; X \overset{\iota}{\leq} Y \wedge \neg(Y \overset{\iota}{\leq} X)$
- How to define the joint random variable of X, Y, written as XY?
  - $Z \overset{\iota}{=} XY \;\Leftrightarrow\; X \overset{\iota}{\leq} Z \wedge Y \overset{\iota}{\leq} Z \wedge \forall U. \left((X \overset{\iota}{\leq} U \wedge Y \overset{\iota}{\leq} U) \to Z \overset{\iota}{\leq} U\right)$
- How to define conditional independence, written as $X \perp\!\!\!\perp Y | Z$?
  - $X \perp\!\!\!\perp Y | Z \;\Leftrightarrow\; \exists U. U \perp\!\!\!\perp XZ \wedge Y \overset{\iota}{\leq} ZU$

Cardinality

- Check X is (at most) a binary random variable (i.e., $|\mathcal{X}| \leq 2$):

$$\mathrm{card}_{\leq 2}(X) \Leftrightarrow \forall U\big(U \overset{\iota}{<} X \rightarrow U \overset{\iota}{=} \emptyset\big)$$

  - Any random variable with strictly less information than X is degenerate
- The condition that $|\mathcal{X}| \leq n$:

$$\mathrm{card}_{\leq n}(X) \Leftrightarrow \forall U\big(U \overset{\iota}{<} X \rightarrow \mathrm{card}_{\leq n-1}(U)\big)$$

$$\mathrm{card}_{\leq 1}(X) \Leftrightarrow (X \overset{\iota}{=} \emptyset)$$

- Define

$$\mathrm{card}_{=n}(X) \Leftrightarrow \mathrm{card}_{\leq n}(X) \wedge \neg\mathrm{card}_{\leq n-1}(X)$$

$$\mathrm{card}_{\geq n}(X) \Leftrightarrow \neg\mathrm{card}_{\leq n-1}(X)$$

Uniformity

- If X, Y, Z are discrete random variables such that any one of them is a function of the other two, and they are pairwise independent, then they are all uniformly distributed over their supports, which have the same size [Zhang and Yeung, 1997]
- The condition that X is uniformly distributed over its support:

$$\mathrm{unif}(X) \Leftrightarrow \exists Y, Z.\, \mathrm{triple}(X, Y, Z),$$

where

$$\begin{aligned}\mathrm{triple}(X, Y, Z) \Leftrightarrow & X \overset{\iota}{\leq} YZ \wedge Y \overset{\iota}{\leq} XZ \wedge Z \overset{\iota}{\leq} XY \\ & \wedge X \perp\!\!\!\perp Y \wedge X \perp\!\!\!\perp Z \wedge Y \perp\!\!\!\perp Z\end{aligned}$$

  - Satisfied when $X, Y \sim \mathrm{Unif}\{0, \ldots, k-1\}$, $Z = X + Y \bmod k$

- Represent $k \in \mathbb{Z}_{>0}$ as a uniform random variable X with $|\mathcal{X}| = k$
- **Equality.** Formula for checking $|\mathcal{X}| = |\mathcal{Y}|$ for uniform X, Y [Li, 2021]:

$$\mathrm{ueq}(X, Y) \Leftrightarrow \exists U_1, U_2, U_3. \quad \mathrm{triple}(X, U_1, U_2) \wedge \mathrm{triple}(Y, U_1, U_3)$$

  - To check for equality against constants:

$$\mathrm{ueq}_n(X) \Leftrightarrow \mathrm{unif}(X) \wedge \mathrm{card}_{=n}(X)$$

- **Multiplication.** Formula for $|\mathcal{X}||\mathcal{Y}| = |\mathcal{Z}|$ for uniform X, Y, Z:

$$\mathrm{uprod}(X, Y, Z) \Leftrightarrow \exists \tilde{X}, \tilde{Y}. \left(\mathrm{ueq}(X, \tilde{X}) \wedge \mathrm{ueq}(Y, \tilde{Y}) \wedge \tilde{X} \perp\!\!\!\perp \tilde{Y} \wedge \tilde{X}\tilde{Y} \overset{\iota}{=} Z\right)$$

- **Comparison.** Formula for $|\mathcal{X}| \le |\mathcal{Y}|$ for uniform X, Y [Li, 2021]:

$$\mathrm{ule}(X, Y) \Leftrightarrow \exists G, \tilde{Y}. \left(\mathrm{uprod}(X, Y, G) \wedge \mathrm{ueq}(Y, \tilde{Y}) \wedge G \overset{\iota}{\le} Y\tilde{Y}\right)$$

  - "$\Leftarrow$": $G \overset{\iota}{\le} Y\tilde{Y} \Rightarrow |\mathcal{G}| \le |\mathcal{Y}||\tilde{\mathcal{Y}}| \Rightarrow |\mathcal{X}||\mathcal{Y}| \le |\mathcal{Y}|^2$
  - "$\Rightarrow$": $X \sim \mathrm{Unif}\{0, \ldots, a-1\}$, $Y \sim \mathrm{Unif}\{0, \ldots, b-1\}$, $G = (X, Y)$, $\tilde{Y} = X + Y \bmod b$

Addition between Integers

- To define addition, the main idea is that if Z is uniform with $|\mathcal{Z}| = |\mathcal{X}| + |\mathcal{Y}|$, then we can partition $\mathcal{Z}$ into two sets with sizes $|\mathcal{X}|, |\mathcal{Y}|$ respectively
  - If $U \in \{0,1\}$ is the indicator of whether Z is in the first set, then $U \sim \mathrm{Bern}(|\mathcal{X}|/(|\mathcal{X}| + |\mathcal{Y}|))$
- The following checks that X, Y, Z are uniform, $|\mathcal{Z}| = |\mathcal{X}| + |\mathcal{Y}|$, and $U \sim \mathrm{Bern}(|\mathcal{X}|/(|\mathcal{X}| + |\mathcal{Y}|))$:

$$\begin{aligned}\mathrm{frac}(X,Y,Z,U) \Leftrightarrow\ & \big(\mathrm{ueq}_2(U) \wedge \mathrm{uprod}(X,U,Z) \wedge \mathrm{uprod}(Y,U,Z)\big) \\ & \vee \exists \tilde{X}, \tilde{Y}. \big(\mathrm{ueq}(X,\tilde{X}) \wedge \mathrm{ueq}(Y,\tilde{Y}) \wedge \mathrm{unif}(Z) \\ & \wedge \mathrm{card}_{=2}(U) \wedge \neg\mathrm{unif}(U) \wedge U \overset{\iota}{\leq} Z \wedge \tilde{X} \perp\!\!\!\perp \tilde{Y} \perp\!\!\!\perp U \wedge Z \overset{\iota}{\leq} \tilde{X}\tilde{Y}U \\ & \wedge \forall V. \big(\mathrm{smi}(Z,V) \rightarrow \mathrm{smi}(\tilde{X}U, V) \vee \mathrm{smi}(\tilde{Y}U, V)\big)\big)\end{aligned}$$

where

$$\begin{aligned}\mathrm{smi}(X,Y) \Leftrightarrow\ & (X \overset{\iota}{=} Y \overset{\iota}{=} \emptyset) \vee \big(Y \overset{\iota}{\leq} X \wedge \mathrm{card}_{=2}(Y) \\ & \wedge \forall U. (U \overset{\iota}{\leq} X \wedge \mathrm{card}_{=4}(U) \rightarrow \neg\exists V. (\mathrm{card}_{\leq 2}(V) \wedge U \overset{\iota}{\leq} YV))\big)\end{aligned}$$

- We then have $\mathrm{usum}(X,Y,Z) \Leftrightarrow \exists U. \mathrm{frac}(X,Y,Z,U)$

Undecidability of FOTPI

Theorem (Li [2022a])

The first-order theory of probabilistic independence is undecidable, i.e., no algorithm can determine whether a statement in FOTPI holds

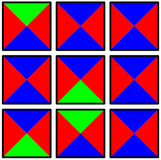
- Direct consequence of the fact that true arithmetic (over natural numbers) is interpretable in FOTPI, and that true arithmetic is undecidable [Tarski, 1933]

Undecidability of CI when one RV is Binary [Li, 2021]

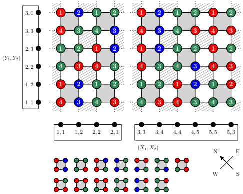
- It is **undecidable** to determine whether

$$|\mathcal{X}_1| \le 2 \wedge \bigwedge_{i=1}^{k} X_{A_i} \perp\!\!\!\perp X_{B_i} | X_{C_i} \Rightarrow X_{A_0} \perp\!\!\!\perp X_{B_0} | X_{C_0}$$

- Use $\mathrm{unif}(X_1)$ to force X_1 to be uniform, and make independent copies
- Use comparison to force any RV to have any cardinality
  - E.g. $a = 5$ is the only solution to $2^9 \le a^4 \le 2^{10}$
- Reduction from periodic tiling problem [Gurevich and Koryakov, 1972]: deciding whether a set of square tiles can tile a torus
- Use uniform RVs to represent coordinates and colors



$\Rightarrow$



Undecidability of CI Implication [Li, 2022b]

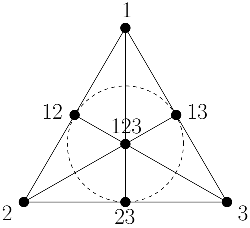
- It is **undecidable** to determine whether

$$\bigwedge_{i=1}^{k} X_{A_i} \perp\!\!\!\perp X_{B_i} | X_{C_i} \Rightarrow X_{A_0} \perp\!\!\!\perp X_{B_0} | X_{C_0}$$

  for given $(A_i)_i, (B_i)_i, (C_i)_i$
- Use the strategy in undecidability of embedded multivalued dependency [Herrmann, 1995]
- Show undecidability by reduction from uniform word problem for finite monoids [Gurevich, 1966]
- Problem – there is no algebraic structure in the RVs X_i!
- Have to impose some algebraic structure using conditional independence

Undecidability of CI Implication [Li, 2022b]

- RVs $A_1, A_2, A_3, A_{12}, A_{13}, A_{23}, A_{123}$
- Impose the "Fano-non-Fano condition":
  - For any three RVs on same solid line, any one is a function of other two
  - Any three RVs not on same solid/dotted line are independent



Lemma

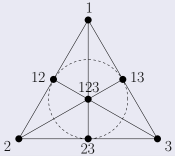
Fano-non-Fano condition holds iff A_1, A_2, A_3 are uniform elements in abelian group, and $A_{12} = A_1 + A_2$, $A_{13} = A_1 + A_3$, $A_{23} = A_2 + A_3$, $A_{123} = A_1 + A_2 + A_3$, up to relabeling

- Equivalent form used in [Herrmann, 1995] for undecidability of EMVD
- Appeared in [Dougherty et al., 2006a] to show unachievability of network coding capacity

Fano-non-Fano Condition

Lemma

Fano-non-Fano condition holds iff A_1, A_2, A_3 are uniform elements in abelian group, $A_{12} = A_1 + A_2$, $A_{13} = A_1 + A_3$, $A_{23} = A_2 + A_3$, $A_{123} = A_1 + A_2 + A_3$, up to relabeling



- A_k is a function of A_i, A_j, let this function be $f_k^{i,j}(a_i, a_j)$
- Bijection between independent (A_i, A_j, A_k) and $(A_i)_i$, let function from (A_i, A_j, A_k) to A_l be $f_l^{i,j,k}(a_i, a_j, a_k)$

Lemma

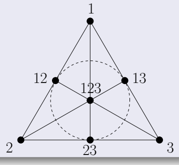
We have

- $f_k^{i,j}(a, b) = f_k^{j,i}(b, a)$, *and* $f_l^{i,j,k}(a, b, c) = f_l^{j,k,i}(b, c, a)$
- $f_i^{k,j}(f_k^{i,j}(a, b),\ b) = a$
- $f_l^{i,j,k}(a, b, c) = f_l^{m,k}(f_m^{i,j}(a, b), c)$

Fano-non-Fano Condition

Lemma

Fano-non-Fano condition holds iff A_1, A_2, A_3 *are uniform elements in abelian group,* $A_{12} = A_1 + A_2$, $A_{13} = A_1 + A_3$, $A_{23} = A_2 + A_3$, $A_{123} = A_1 + A_2 + A_3$, *up to relabeling*



- A_k is a function of A_i, A_j, let this function be $f_k^{i,j}(a_i, a_j)$
- Bijection between independent (A_i, A_j, A_k) and $(A_i)_i$, let function from (A_i, A_j, A_k) to A_l be $f_l^{i,j,k}(a_i, a_j, a_k)$
- $f_{12}^{1,2} = f_{12}^{2,1} = f_{13}^{1,3} = f_{13}^{3,1} = f_{23}^{2,3} = f_{23}^{3,2} = f_{123}^{1,23} = f_{123}^{2,13} = f_{123}^{3,12} = f_{123}^{23,1} = f_{123}^{13,2} = f_{123}^{12,3}$
- Define abelian group over $\mathcal{A}$ by $a + b := f_{12}^{1,2}(a, b)$, $-a := f_2^{1,12}(a, 0)$

Undecidability of CI Implication

- Strategy proposed by Dougherty [2009] – reduction from the identity problem for finite groups
  - Identity – equality that holds **for all** values of the variables, e.g., $\forall x, y.\, xy = yx$ (iff group is abelian)
  - Identity problem – whether a list of identities implies another identity

$$\bigwedge_{i=1}^{l} \left(\forall x_{1..k}.P_i(x_{1..k})\right) \;\rightarrow\; \forall x_{1..k}.P_0(x_{1..k})$$

  - Uniform RVs act as the universally-quantified variables
  - However, identity problem for finite groups is not known to be decidable or undecidable [Albert et al., 1992]!
- Herrmann [1995], Li [2022b]: instead use uniform word problem for finite monoids [Gurevich, 1966]
  - Whether a list of equalities implies another equality

$$\forall x_{1..k}.\left(\bigwedge_{i=1}^{l} P_i(x_{1..k}) \;\rightarrow\; P_0(x_{1..k})\right)$$

  - Need to use uniform RVs to represent specific monoid elements

Word Problem and Endomorphism Monoid

- Uniform word problem for finite monoids [Gurevich, 1966] – Given $a_i, b_i, c_i \in \{1, \ldots, k\}$ for $i = 0, \ldots, l$, determine whether the implication

$$\bigwedge_{i=1}^{l} (x_{a_i} \cdot x_{b_i} = x_{c_i}) \ \rightarrow \ (x_{a_0} = x_{c_0})$$

  holds for all finite monoids $\mathcal{M}$ and all k-tuples $x_1, \ldots, x_k \in \mathcal{M}$
- Consider endomorphism monoid of abelian group
  - Homomorphism $g : \mathcal{A} \rightarrow \mathcal{B}$ between abelian groups $\mathcal{A}, \mathcal{B}$ is a function satisfying $g(a + b) = g(a) + g(b)$
  - Endomorphism in $\mathcal{A}$ is a homomorphism $g : \mathcal{A} \rightarrow \mathcal{A}$
  - The *endomorphism monoid* $\mathrm{End}(\mathcal{A})$ is the set of endomorphisms in $\mathcal{A}$, equipped with the operation $g \cdot h : \mathcal{A} \rightarrow \mathcal{A}$ where $g \cdot h(a) = g(h(a))$
- Kurosh [1963] – For any finite monoid, there exists embedding from that monoid into $\mathrm{End}(\mathcal{A})$ for some finite abelian group $\mathcal{A}$
  - No loss of generality of considering only endomorphism monoids

Representing Endomorphism by RV

- A_1, A_2, A_3 are uniform elements in abelian group $\mathcal{A}$, and $A_{12} = A_1 + A_2$, $A_{13} = A_1 + A_3$, $A_{23} = A_2 + A_3$, $A_{123} = A_1 + A_2 + A_3$
- Represent an endomorphism $g : \mathcal{A} \to \mathcal{A}$ by $U = A_1 - g(A_2)$
- Check whether U corresponds to an endomorphism [Herrmann, 1995, Li, 2023]:

$$\begin{aligned} \mathrm{end}_{1,2}((A_i)_i, U) \;\Leftrightarrow\; \exists V, W :\; & \mathrm{FanoNonFano}((A_i)_i) \wedge \mathrm{ueq}(U, A_1) \\ & \wedge\ \mathrm{ueq}(V, A_1) \wedge \mathrm{ueq}(W, A_1) \wedge U \overset{\iota}{=} A_1 | A_2 \\ & \wedge\ V \overset{\iota}{=} A_1 | A_{23} \wedge U \overset{\iota}{=} V | A_3 \wedge W \overset{\iota}{=} A_{13} | A_2 \wedge U \overset{\iota}{=} W | A_3, \end{aligned}$$

  where $X \overset{\iota}{=} Y|Z \;\Leftrightarrow\; X \overset{\iota}{\leq} ZY \wedge Y \overset{\iota}{\leq} ZX$, i.e., if we are given Z, then X has the same information as Y, and $\mathrm{ueq}(X, Y)$ checks whether X, Y are both uniform and have the same cardinality
  - "$\Rightarrow$": $V = A_1 - g(A_2 + A_3)$, $W = A_1 - g(A_2) + A_3$
- Representing composition – if $\mathrm{end}_{1,2}((A_i)_i, U_1)$, $\mathrm{end}_{2,3}((A_i)_i, U_2)$, $\mathrm{end}_{1,3}((A_i)_i, U_3)$, we have $U_3 \overset{\iota}{\leq} U_1 U_2$ iff $g_3 = g_1 \cdot g_2$
  - "$\Leftarrow$": $U_3 = A_1 - g_1 \cdot g_2(A_3) = A_1 - g_1(A_2) + g_1(A_2 - g_2(A_3))$

Representing Endomorphism by RV

- A_1, A_2, A_3 are uniform elements in abelian group $\mathcal{A}$, and $A_{12} = A_1 + A_2$, $A_{13} = A_1 + A_3$, $A_{23} = A_2 + A_3$, $A_{123} = A_1 + A_2 + A_3$
- Represent an endomorphism $g : \mathcal{A} \to \mathcal{A}$ by $U = A_1 - g(A_2)$
- Check whether U corresponds to an endomorphism [Herrmann, 1995, Li, 2023]: $\mathrm{end}_{1,2}((A_i)_{i\in\mathcal{E}}, U)$
- Representing composition – if $\mathrm{end}_{1,2}((A_i)_i, U_1)$, $\mathrm{end}_{2,3}((A_i)_i, U_2)$, $\mathrm{end}_{1,3}((A_i)_i, U_3)$, we have $U_3 \overset{\iota}{\le} U_1 U_2$ iff $g_3 = g_1 \cdot g_2$
- Need to convert $\mathrm{end}_{2,3}, \mathrm{end}_{1,3}$ to $\mathrm{end}_{1,2}$
- Convert $\mathrm{end}_{i,j}$ for different i, j:

$$\begin{aligned}\mathrm{conv}^{1,2}_{1,3}((A_i)_i, U, V) \Leftrightarrow \exists W :\ & \mathrm{end}_{1,2}((A_i)_i, U) \\ & \wedge\ \mathrm{end}_{1,3}((A_i)_i, V) \wedge \mathrm{end}_{2,3}((A_i)_i, W) \\ & \wedge\ A_{13} \overset{\iota}{\le} A_{12} W \wedge V \overset{\iota}{\le} UW\end{aligned}$$

Representing Endomorphism by RV

- A_1, A_2, A_3 are uniform elements in abelian group $\mathcal{A}$, and $A_{12} = A_1 + A_2$, $A_{13} = A_1 + A_3$, $A_{23} = A_2 + A_3$, $A_{123} = A_1 + A_2 + A_3$
- Represent an endomorphism $g : \mathcal{A} \to \mathcal{A}$ by $U = A_1 - g(A_2)$
- Check whether U corresponds to an endomorphism [Herrmann, 1995, Li, 2023]: $\mathrm{end}_{1,2}((A_i)_{i\in\mathcal{E}}, U)$
- Representing composition – if $\mathrm{end}_{1,2}((A_i)_i, U_1)$, $\mathrm{end}_{2,3}((A_i)_i, U_2)$, $\mathrm{end}_{1,3}((A_i)_i, U_3)$, we have $U_3 \overset{\iota}{\le} U_1 U_2$ iff $g_3 = g_1 \cdot g_2$
- Convert $\mathrm{end}_{i,j}$ for different i, j: $\mathrm{conv}_{1,3}^{1,2}((A_i)_i, U, V)$
- Check whether U_1, U_2, U_3 with $\mathrm{end}_{1,2}((A_i)_i, U_j)$ satisfy $g_3 = g_1 \cdot g_2$:

$$
\begin{aligned}
&\mathrm{comp}_{1,2}((A_i)_i, U_1, U_2, U_3) \;\Leftrightarrow\\
&\exists V_1, V_2 : \bigwedge_{j=1}^{3} \mathrm{end}_{1,2}((A_i)_i, U_j) \wedge \mathrm{conv}_{1,3}^{1,2}((A_i)_i, U_1, V_1)\\
&\wedge \mathrm{conv}_{3,2}^{1,2}((A_i)_i, U_2, V_2) \wedge U_3 \overset{\iota}{\le} V_1 V_2
\end{aligned}
$$

Undecidability of CI Implication [Li, 2022b]

- Uniform word problem for finite monoids [Gurevich, 1966]

$$\bigwedge_{i=1}^{l} (x_{a_i} \cdot x_{b_i} = x_{c_i}) \;\rightarrow\; (x_{a_0} = x_{c_0})$$

holds for all finite monoid $\mathcal{M}$ and all k-tuples $x_1, \ldots, x_k \in \mathcal{M}$

is true iff...

-

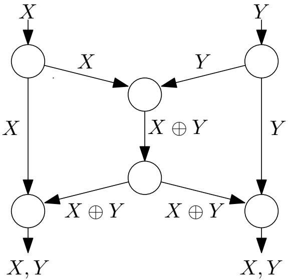
$$\left(\bigwedge_{j=1}^{k} \mathrm{end}_{1,2}((A_i)_i, U_j) \;\wedge\; \bigwedge_{j=1}^{l} \mathrm{comp}_{1,2}((A_i)_i, U_{a_j}, U_{b_j}, U_{c_j}) \right)$$
$$\rightarrow\; (U_{a_0} \overset{\iota}{\leq} U_{c_0})$$

holds for all finite random variables $(A_i)_i$, $U_1, \ldots, U_k$

- Since uniform word problem for finite monoids is undecidable, CI implication is undecidable as well

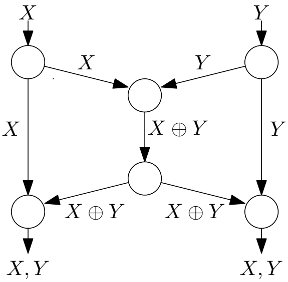
- Sequence of random variables $X^n = (X_1, \ldots, X_n)$
- Entropic vector [Zhang and Yeung, 1997] $\mathbf{h}(X^n) = \mathbf{h} \in \mathbb{R}^{2^n-1}$, where entries of $\mathbf{h}$ are indexed by nonempty subsets of $[n]$, and $\mathbf{h}_S := H(X_S)$
- Entropic region $\Gamma_n^* := \bigcup_{p_{X^n}} \{\mathbf{h}(X^n)\}$ [Zhang and Yeung, 1997]
- Non-Shannon inequalities (cannot be deduced from $I(X; Y|Z) \geq 0$) were given in [Zhang and Yeung, 1998, Makarychev et al., 2002, Dougherty et al., 2006b]
- Matúš [2007] showed that $\overline{\Gamma_n^*}$ is not polyhedral
- Conditional information inequalities: whether a linear inequality follows from a list of inequalities
  - Can encode conditional independence implication, and hence **undecidable** [Li, 2022b]
- Decidability of unconditional information inequalities is open

Related Problems: Network Coding



- Network coding [Ahlswede et al., 2000, Li et al., 2003]
  - Network of nodes connected by noiseless links with same capacity
  - Each source node has a message, and each destination node desires a set of messages
  - Each node is capable of performing coding, not only routing
- If there is one source and multiple destinations, the capacity (number of message bits per link capacity) is given by the maximum network flow [Ahlswede et al., 2000]
  - Single-source multicast network coding is **decidable**
- Significantly harder if there are multiple sources

Related Problems: Network Coding



- NP-hardness results: Lehman [2005], Langberg et al. [2006], Langberg and Sprintson [2011]
- Is network coding decidable?
  - Given a network, if the message size and the link capacity are the same, does there exist a valid coding scheme?
  - Partial result: whether a network admits a vector linear network code is undecidable [Kühne and Yashfe, 2019]
  - Shown to be **undecidable** in [Li, 2022b]
- Decidability of whether the capacity can be approached is unknown

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